13.1. Harmonic function in the disk

Let $D := \{x^2 + y^2 < 1\}$. Find the solution to the following problem

$$\begin{cases} \Delta u = 0, & \text{for } (x,y) \in D, \\ u(x,y) = x^3 + x, & \text{for } (x,y) \in \partial D. \end{cases}$$

Hint: It holds $\cos(\theta)^3 = \frac{1}{4}(3\cos(\theta) + \cos(3\theta))$.

Let us consider the polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$. Let us begin, by writing the boundary condition in polar coordinates and exploiting the hint

$$u(x,y) = x^3 + x = \cos(\theta)^3 + \cos(\theta) = \frac{1}{4}(3\cos(\theta) + \cos(3\theta)) + \cos(\theta) = \frac{7}{4}\cos(\theta) + \frac{1}{4}\cos(3\theta)\,.$$

Since $r\cos(\theta)$ and $r^3\cos(3\theta)$ are harmonic functions in the unit disk D, we deduce that

$$u = \frac{7}{4} r\cos(\theta) + \frac{1}{4} r^3 \cos(3\theta)$$

is harmonic and satisfies the boundary condition, hence, by uniqueness, it must be the only solution of the problem.

13.2. Harmonic function in the annulus

Find the solution to the following problem, posed for $2 < r < 4$ and $-\pi < \theta \leq \pi$:

$$\begin{cases} \Delta u = 0, & \text{for } 2 < r < 4, \\ u(2,\theta) = 0, & \text{for } -\pi < \theta \leq \pi, \\ u(4,\theta) = \sin(\theta), & \text{for } -\pi < \theta \leq \pi. \end{cases}$$

We do separation of variables in polar coordinates. Namely, we express a general solution $w(r, \theta) = R(r)\Theta(\theta)$, and we assume $\Delta w = 0$. Recall that the Laplacian in polar coordinates can be written as

$$\Delta w = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0.$$

Thus, in the annulus $\{2 < r < 4\}$ we have that

$$0 = \Delta w = R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta''.$$

That is, dividing by $\frac{1}{r^2} R\Theta$, and redistributing the terms, we have that

$$-\frac{\Theta''}{\Theta} = r^2 \frac{R''}{R} + r\frac{R'}{R} = \lambda \in \mathbb{R}.$$

That is, both sides are constant. We reach the equations

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0,$$

for $2 < r < 4$, and

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0,$$

for $-\pi < \theta \leq \pi$. From the periodicity assumptions, we know that the solution Θ must fulfil $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$. This directly implies that the solutions for Θ are of the form

$$\Theta_n(\theta) = \alpha_n \cos(n\theta) + \beta_n \sin(n\theta),$$

with $\lambda_n = n^2$ and $n \geq 0$. We now want to solve the equation for R, to find R_n such that

$$r^2R_n''(r) + rR_n'(r) - n^2R_n(r) = 0.$$

By taking the guess that solutions are of the form r^α for some α, we reach that two possible solutions to the previous equation for $n \geq 1$ are r^n and r^{-n} (up to multiplicative constants)[1]. Thus, we have that the general solution to the previous equation is given, for $n \geq 1$ is given by

$$R_n(r) = \gamma_n r^n + \delta_n r^{-n},$$

for some constants γ_n and δ_n. If $n = 0$, then the general solution is easily obtained to be

$$R_0(r) = \gamma_0 + \delta_0 \log(r).$$

Thus, we are looking for a general solution of the form

$$\begin{aligned} u(r,\theta) = A_0 + B_0\log(r) &+ \sum_{n\geq 1} r^n(A_n\cos(n\theta) + B_n\sin(n\theta)) \\ &+ \sum_{n\geq 1} r^{-n}(C_n\cos(n\theta) + D_n\sin(n\theta)), \end{aligned}$$

for some constants A_n, B_n (for $n \geq 0$) and C_n, D_n (for $n \geq 1$) to be determined.

[1] That is, if $n \geq 1$, we guess that the solution is of the form $R_n(r) = Cr^\alpha$ for some constant. Plugging into the equation, this means that

$$0 = r^2R_n''(r) + rR_n'(r) - n^2R_n(r) = r^2\alpha(\alpha-1)Cr^{\alpha-2} + r\alpha Cr^{\alpha-1} - n^2Cr^\alpha.$$

Rearranging terms we get that $Cr^\alpha\left(\alpha^2 - n^2\right) = 0$, which holds if $\alpha = \pm n$. Thus, Cr^n and Cr^{-n} are both admissible solutions. A second order linear ODE has a two-dimensional space of solutions, therefore, our solutions will be linear combinations of r^n and r^{-n} .

A similar argument gives the solutions in the case $n = 0$.

Notice that, since the point $r = 0$ is not included in the domain, it makes sense to consider the negative powers r^{-n} (as well as $\log(r)$) as possible solutions to our equation. Imposing the boundary conditions, we get that

$$\begin{aligned} 0 = u(2,\theta) = A_0 + B_0 \log(2) &+ \sum_{n\geq 1} 2^n (A_n \cos(n\theta) + B_n \sin(n\theta)) \\ &+ \sum_{n\geq 1} 2^{-n} (C_n \cos(n\theta) + D_n \sin(n\theta)), \end{aligned}$$

On the other hand,

$$\begin{aligned} \sin(\theta) = u(4,\theta) = A_0 + B_0 \log(4) &+ \sum_{n\geq 1} 4^n (A_n \cos(n\theta) + B_n \sin(n\theta)) \\ &+ \sum_{n\geq 1} 4^{-n} (C_n \cos(n\theta) + D_n \sin(n\theta)), \end{aligned}$$

In particular, $A_0 + B_0 \log(2) = A_0 + B_0 \log(4) = 0$ so that $A_0 = B_0 = 0$. On the other hand, for $n \geq 2$, $2^n A_n + 2^{-n} C_n = 4^n A_n + 4^{-n} C_n = 0$, so that $A_n = C_n = 0$. Similarly, if $n \geq 2$, $B_n = D_n = 0$. And to finish, we notice that

$$2B_1 + 2^{-1} D_1 = 0, \qquad 4B_1 + 4^{-1} D_1 = 1,$$

from where we deduce that $D_1 = -\frac{4}{3}$ and $B_1 = \frac{1}{3}$. That is, our solution is given by

$$u(r,\theta) = r\frac{\sin(\theta)}{3} - \frac{4\sin(\theta)}{3r}.$$

Alternative solution: We could directly notice that the boundary values depend only on $\sin(\theta)$, in order to find an expression involving only this terms. That is, we could guess that $u(r,\theta)$ is of the form

$$u(r,\theta) = B_1 r \sin(\theta) + D_1 r^{-1} \sin(\theta),$$

and compute the values of B_1 and D_1 from the boundary conditions as before. This gives

$$u(r,\theta) = r\frac{\sin(\theta)}{3} - \frac{4\sin(\theta)}{3r},$$

which fulfils the problem. Moreover, by uniqueness, since u is a solution, is the only solution.

13.3. Big on the boundary, small inside

Let $B_r := \{x^2 + y^2 < r\}$ be the ball centered at the origin with radius $r > 0$. Find a harmonic function $u : \bar{B}_1 \to \mathbb{R}$ such that

$$|u| < 0.00001 \text{ in } B_{\frac{1}{2}} \quad \text{and} \quad \int_{\partial B_1} |u| > 1000\,.$$

Let us consider the polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$. Let $u := Nr^N \sin(N\theta)$, where $N = 1000$. The function u is harmonic.

We have

$$\int_{\partial B_1} |u| = N \int_0^{2\pi} |\sin(N\theta)|\, d\theta = 4N = 4000 > 1000\,.$$

Moreover, if $(x, y) \in B_{\frac{1}{2}}$ and (r, θ) is the polar representation of (x, y), then $r < \frac{1}{2}$. Hence, for $(x, y) \in B_{\frac{1}{2}}$, it holds

$$|u(x, y)| = Nr^N |\sin(N\theta)| \le N\frac{1}{2^N} = 1000 \cdot 2^{-1000} < 0.00001\,.$$