10.1. Unique solution

Let $k > 0$. Let D be a bounded planar domain in $\mathbb{R}^2$. Let $u = u(x, y)$ be a solution to the Dirichlet problem for the reduced Helmholtz energy in D. That is, let u solve

$$\begin{cases} \Delta u(x,y) - ku(x,y) &= 0, \quad \text{for } (x,y) \in D, \\ u(x,y) &= g(x,y), \quad \text{for } (x,y) \in \partial D. \end{cases}$$

Show that there exists at most a unique solution twice differentiable in D and continuous in $\overline{D}$, that is, $u \in C^2(D) \cap C(\overline{D})$.

Hint: Assume that there exist two solutions u_1 and u_2, and consider the difference $v = u_1 - u_2$.

SOL:

Let us use the hint. Let us suppose that there exist two solutions u_1 and u_2 fulfilling the Dirichlet problem. Let $v = u_1 - u_2$. Notice that $v = v(x, y)$ solves

$$\begin{cases} \Delta v(x,y) - kv(x,y) &= 0, \quad \text{for } (x,y) \in D, \\ v(x,y) &= 0, \quad \text{for } (x,y) \in \partial D. \end{cases}$$

We just need to prove that $v \equiv 0$ in D. To do so, we will show that $\max_{\overline{D}} v = \min_{\overline{D}} v = 0$. We show both equalities by contradiction.

Notice that $\max_{\overline{D}} v \geq 0$, since $v = 0$ on ∂D. Let us suppose that $\max_{\overline{D}} v = M > 0$. In particular, there exists some $(x_\circ, y_\circ) \in D$ such that $v(x_\circ, y_\circ) = M > 0$, that is, v has a maximum at $(x_\circ, y_\circ)$. In particular, we know that $\Delta v(x_\circ, y_\circ) \leq 0$. Therefore,

$$0 = \Delta v(x_\circ, y_\circ) - kv(x_\circ, y_\circ) \leq -kM < 0,$$

a contradiction.

On the other hand, $\min_{\overline{D}} v \leq 0$, since $v = 0$ on ∂D. Let us suppose that $\min_{\overline{D}} v = m < 0$. In particular, there exists some $(x_\circ, y_\circ) \in D$ such that $v(x_\circ, y_\circ) = m < 0$, that is, v has a minimum at $(x_\circ, y_\circ)$. In particular, we know that $\Delta v(x_\circ, y_\circ) \geq 0$. Therefore,

$$0 = \Delta v(x_\circ, y_\circ) - kv(x_\circ, y_\circ) \geq -km > 0,$$

a contradiction. Therefore, if there exists a solution, is unique.

10.2. The mean-value principle Let D be a planar domain, and let $B_R((x_\circ, y_\circ))$ (ball of radius R centered at $(x_\circ, y_\circ)$) be fully contained in D. Let u be an harmonic function in D, $\Delta u = 0$ in D. Then, the mean-value principle says that the value of u at $(x_\circ, y_\circ)$ is the average value of u on $\partial B_R((x_\circ, y_\circ))$. That is,

$$u(x_\circ, y_\circ) = \frac{1}{2\pi R} \oint_{\partial B_R((x_\circ, y_\circ))} u(x(s), y(s))\, ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_\circ + R\cos\theta, y_\circ + R\sin\theta)\, d\theta.$$

Show that $u(x_\circ, y_\circ)$ is also equal to the average of u in $B_R((x_\circ, y_\circ))$, that is,

$$u(x_\circ, y_\circ) = \frac{1}{\pi R^2} \int_{B_R((x_\circ, y_\circ))} u(x, y)\, dx\, dy.$$

SOL:

Let us use polar coordinates to compute

$$\begin{aligned}
\frac{1}{\pi R^2} \int_{B_R((x_\circ, y_\circ))} u(x, y)\, dx\, dy &= \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} u(x_\circ + r\cos\theta, y_\circ + r\sin\theta) r\, d\theta\, dr \\
&= \frac{1}{\pi R^2} \int_0^R r \left(\int_0^{2\pi} u(x_\circ + r\cos\theta, y_\circ + r\sin\theta)\, d\theta \right) dr \\
&= \frac{1}{\pi R^2} \int_0^R 2\pi r u(x_\circ, y_\circ)\, dr \\
&= u(x_\circ, y_\circ) \frac{1}{\pi R^2} [\pi r^2]_0^R \\
&= u(x_\circ, y_\circ).
\end{aligned}$$

We have used here the boundary mean value principle in the balls $B_r((x_\circ, y_\circ))$ for each $r \in (0, R)$.

10.3. Maximum principle Consider the disk $D := \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\right\}$. Let $u = u(x, y)$ be a function twice differentiable in D and continuous in $\bar{D}$, solving

$$\begin{cases} \Delta u(x, y) = 0, & \text{in } D, \\ u(x, y) = g(x, y), & \text{on } \partial D, \end{cases}$$

for some given function g.

(a) Suppose $g(x, y) = x^2 + \frac{2}{\sqrt{2}} y$. Compute $u(0, 0)$ and $\max_{(x,y)\in\bar{D}} u(x, y)$.

(b) Suppose now that g is any smooth function such that $g(x, y) \geq (3x - y)$. Show that $u(1/3, 0) \geq 1$, with equality if and only if $g(x, y) = 3x - y$.

Hint: the function $3x - y$ is harmonic.

SOL:

(a) By the mean value property

$$u(0,0) = \frac{1}{2\pi}\int_{\partial D} g\,dl = \frac{1}{2\pi}\int_0^{2\pi} \cos(\theta)^2 + \frac{2}{\sqrt{2}}\sin(\theta)\,d\theta = \frac{\pi}{2\pi} = \frac{1}{2}.$$

By the Maximum Principle

$$\max_{\bar{D}} u = \max_{\partial D} u = \max_{\partial D} g = \max_{\theta\in[0,2\pi)} \{\cos(\theta)^2 + \frac{2}{\sqrt{2}}\sin(\theta)\}.$$

Setting $g(\theta) = \cos^2(\theta) + \frac{2}{\sqrt{2}}\sin(\theta)$, we have that (up to periodicity)

$$g'(\theta) = \cos(\theta)(\frac{2}{\sqrt{2}} - 2\sin(\theta)) = 0,$$

if and only if $\theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}\}$. A quick check shows that $\max_\theta g(\theta) = g(\pi/4) = \frac{3}{2}$.

(b) It is convenient to set the auxiliary function $w := u - 3x + y$. Then

$$\begin{cases} \Delta w = 0, & \text{in } D \\ w = g - (3x - y) \geq 0, & \text{on } \partial D, \end{cases}$$

by the very assumption on g. Applying the Maximum Principle to w, we get that

$$\min_{\bar{D}}(u - (3x - y)) = \min_{\bar{D}} w = \min_{\partial D} w \geq 0,$$

implying that $u(x,y) \geq 3x - y$ in $\bar{D}$. In particular,

$$u(1/3,0) \geq 3 \cdot \frac{1}{3} = 1.$$

If $u(1/3,0) = 1$, then w attains its minimum in D since $w(1/3,0) = u(1/3,0) - 1 = 0$. This implies by the strong maximum principle that $w \equiv 0$, and hence $u(x,y) = 3x - y$. In particular, $g(x,y) = u(x,y) = 3x - y$ on ∂D. This shows the 'only if' direction. The 'if' direction is a consequence of uniqueness of solution of the Laplace equations with Dirichlet boundary condition on w.

10.4. Multiple choice Cross the correct answer(s).

(a) Consider the Neumann problem for the Poisson equation

$$\begin{cases} \Delta u = \rho, & \text{in } D, \\ \partial_\nu u = g, & \text{on } \partial D, \end{cases}$$

where $D = B(0, R)$ is the ball of radius $R > 0$ with centre in the origin of $\mathbb{R}^2$, and ρ and g are given in polar coordinates (r, θ) by

$$\rho(r, \theta) = r^\alpha \sin^2(\theta), \text{ and } g(r, \theta) = C \cos^2(\theta) + r^{2021} \sin(\theta),$$

for some constants $\alpha > 0$ and $C > 0$. For which values of $C > 0$ does the problem satisfy the Neumann's *necessary* condition for existence of solutions?

- X $C = \frac{R^{\alpha+1}}{\alpha+2}$
- ◯ $C = \frac{R^{\alpha+2}}{\alpha+2}$
- ◯ $C = \frac{R^{\alpha+1}}{\alpha+1}$
- ◯ $C = \frac{R^{\alpha+1}}{\alpha-1}$

SOL: We say that the Neumann Problem for the Poisson equation satisfies the necessary condition for existence of solutions if the identity

$$\int_{\partial D} g = \int_D \rho, \tag{1}$$

holds. In our particular case we can compute in polar coordinates

$$\int_D \rho = \int_0^R r \int_0^{2\pi} r^\alpha \sin^2(\theta)\, d\theta\, dr = \pi \frac{R^{\alpha+2}}{\alpha + 2},$$

and parametrizing ∂D with the curve $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ we have that

$$\int_{\partial D} g = \int_0^{2\pi} R\Big(C\cos^2(\theta) + R^{2021}\sin(\theta)\Big)\, d\theta = RC\pi.$$

Plugging this in Equation (1) we obtain that the identity

$$RC\pi = \pi \frac{R^{\alpha+2}}{\alpha + 2},$$

is valid if and only if $C = \frac{R^{\alpha+1}}{\alpha+2}$.

(b) Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = \frac{x}{x^2+y^2} & \text{on } \partial D, \end{cases}$$

where the domain D is the anulus defined by $D := \Big\{(x, y) \in \mathbb{R}^2 : 1 < \sqrt{x^2 + y^2} < 2\Big\}$. What is the maximum of u?

○ $\frac{1}{2}$ ○ $\frac{1}{4}$

X 1 ○ -1

SOL: By the weak maximum principle, $\max_{\bar{D}} u = \max_{\partial D} u = \max_{\partial D} \frac{x}{x^2+y^2}$. Writing $\partial D = \{x^2+y^2=1\} \cup \{x^2+y^2=4\} =: S^1 \cup S^2$, we check that $\max_{S^1} u = \max_{S^1} x = 1$, and $\max_{S^2} u = \max_{S^2} \frac{x}{4} = \frac{1}{2}$. Hence $\max_{\partial D} u = \max\{1, \frac{1}{2}\} = 1$.

Extra exercises

10.5. Weak maximum principle Let B_1 denote the unit ball in $\mathbb{R}^2$ centered at the origin, and let $u = u(x,y)$ be twice differentiable in B_1 and continuous in $\overline{B_1}$. Suppose that u solves the Dirichlet problem

$$\begin{cases} \Delta u(x,y) &= -1, & \text{for } (x,y) \in B_1, \\ u(x,y) &= g(x,y), & \text{for } (x,y) \in \partial B_1. \end{cases}$$

Show that

$$\max_{\bar{B}_1} u \leq \frac{1}{2} + \max_{\partial B_1} g.$$

Hint: search for a simple *function w such that $\Delta w = 1$, and use it to reduce the problem to an application of the weak maximum principle for harmonic functions.*

SOL:

We just need to find a function $w(x,y)$ such that $\Delta w(x,y) = 1$, and then consider $v(x,y) = u(x,y) + w(x,y)$. The simplest function such that $\Delta w(x,y) = 1$ is $w(x,y) = \frac{1}{2}x^2$. Thus, let us define

$$v(x,y) = u(x,y) + \frac{1}{2}x^2.$$

Then, v solves

$$\begin{cases} \Delta v(x,y) &= 0, & \text{for } (x,y) \in B_1, \\ v(x,y) &= g(x,y) + \frac{1}{2}x^2, & \text{for } (x,y) \in \partial B_1. \end{cases}$$

By the weak maximum principle, we know that

$$\max_{\bar{B}_1} v(x,y) = \max_{\partial B_1} \left(g(x,y) + \frac{1}{2}x^2 \right) \leq \max_{\partial B_1} g(x,y) + \max_{\partial B_1} \frac{1}{2}x^2.$$

Notice that $\max_{\partial B_1} \frac{1}{2}x^2 = \frac{1}{2}$, so

$$\max_{\bar{B}_1} v(x,y) \leq \frac{1}{2} + \max_{\partial B_1} g(x,y).$$

On the other hand, $v(x, y) \geq u(x, y)$ for all $x, y \in B_1$, so

$$\max_{\bar{B}_1} u(x, y) \leq \max_{\bar{B}_1} v(x, y) \leq \frac{1}{2} + \max_{\partial B_1} g(x, y),$$

as we wanted to see.