9.1. Separation of variables for non-homogeneous problems

Solve the following equations using the method of separation of variables and superposition principle. If the boundary conditions are non-homogeneous, find a suitable function satisfying the boundary conditions, and subtract it from the solution.

(a)

$$\begin{cases} u_t - u_{xx} &= t + 2\cos(2x), & (x,t) \in (0,\pi/2) \times (0,\infty), \\ u_x(0,t) &= 0, & t \in (0,\infty), \\ u_x(\pi/2,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= 1 + 2\cos(6x), & x \in [0,\pi/2]. \end{cases}$$

(b)

$$\begin{cases} u_t - u_{xx} &= 1 + x\cos(t), & (x,t) \in (0,1) \times (0,\infty), \\ u_x(0,t) &= \sin(t), & t \in (0,\infty), \\ u_x(1,t) &= \sin(t), & t \in (0,\infty), \\ u(x,0) &= 1 + \cos(2\pi x), & x \in [0,1]. \end{cases}$$

Hint: The function $w(x,t) = x\sin(t)$ fulfills the boundary conditions from above.

(c) Mixed Boundary Conditions.

$$\begin{cases} u_t - u_{xx} &= \sin(9x/2), & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u_x(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(3x/2), & x \in [0,\pi]. \end{cases}$$

SOL:

(a) In Lecture 9 we have seen that the solution of the inhomogeneous problem can be obtained by applying the method of separation of variables, that is writing

$$u(x,t) = \sum_{n\geq 0} T_n(t) X_n(x),$$

where the ODE solved by $X_n(x)$ and $T_n(t)$ are determined by the homogeneous problem (i.e. setting 0 to the right hand side of the PDE). We already know that the function $v(x,t) = X(x)T(t)$ solves the homogeneous problem $v_t - v_{xx} = 0$ if $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \lambda = \text{constant}$, which implies as usual that

- If $\lambda > 0$,

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

- If $\lambda = 0$,
$$X(x) = A + Bx.$$
- If $\lambda < 0$,
$$X(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x).$$

Imposing the initial conditions $v_x(0,t) = v_x(\pi/2,t) = 0$ is equivalent to ask $X'(0) = X'(\pi/2) = 0$. This implies directly that $A = B = 0$ if $\lambda < 0$, and $B = 0$ if $\lambda = 0$. If $\lambda > 0$ we get that $B = 0$, and $\cos(\sqrt{\lambda}\pi/2) = 0$, that is, $\lambda_n = 4n^2$ is the set of possible values for λ (including $n = 0$ to account for the constant arising from $\lambda = 0$). Thus, the corresponding solutions are $X_n(x) = \cos(2nx)$, and we are looking for a general solution of the form
$$u(x,t) = \sum_{n\geq 0} T_n(t)\cos(2nx)$$
for some functions $T_n(t)$ to be determined. From the initial condition, we directly get that
$$T_0(0) = 1, \quad T_3(0) = 2, \quad T_n(0) = 0, \text{ for all } n \notin \{0,3\}.$$

On the other hand, imposing that $u_t - u_{xx} = t + 2\cos(2x)$ we get
$$\sum_{n\geq 0}(T_n'(t) + 4n^2T_n(t))\cos(2nx) = t + 2\cos(2x).$$

That is,
$$T_0'(t) = t, \quad T_1'(t) + 4T_1(t) = 2, \quad T_n'(t) + 4n^2(t) = 0, \text{ for all } n \geq 2.$$

We have various ODE with the corresponding initial conditions to be solved for each T_n:

$(n = 0)$ In this case,
$$T_0'(t) = t, \quad T_0(0) = 1, \quad \Rightarrow \quad T_0(t) = 1 + \frac{1}{2}t^2.$$

$(n = 1)$ In this case,
$$T_1'(t) + 4T_1(t) = 2, \quad T_1(0) = 0, \quad \Rightarrow \quad T_1(t) = \frac{1}{2} - \frac{1}{2}e^{-4t}.$$

(To solve the ODE, we notice that the solution to the homogeneous ODE is Ce^{-4t}, and that a particular solution is simply the constant $\frac{1}{2}$. By adding them up, and choosing C such that the initial condition holds, we get our solution.)

$(n = 3)$ In this case,

$$T_3'(t) + 36T_3(t) = 0, \quad T_3(0) = 2, \quad \Rightarrow \quad T_3(t) = 2e^{-36t}.$$

$(n \notin \{0, 1, 3\})$ In this case,

$$T_n'(t) + 4n^2 T_n(t) = 0, \quad T_n(0) = 0, \quad \Rightarrow \quad T_n(t) = 0.$$

Thus, the general solution is given by

$$u(x,t) = 1 + \frac{1}{2}t^2 + \frac{1}{2}\left(1 - e^{-4t}\right)\cos(2x) + 2e^{-36t}\cos(6x).$$

(b) The first thing to notice is that the boundary conditions are now non-homogeneous. Thus, we have to find a new function $w(x,t)$ satisfying such non-homogeneous boundary conditions, and study the problem being satisfied by $v(x,t) = u(x,t) - w(x,t)$.

In this case, from the hint $w(x,t) = x\sin(t)$ satisfies the boundary conditions for $t \geq 0$. Let us write the problem satisfied by $v(x,t) = u(x,t) - x\sin(t)$:

$$\begin{cases} v_t - v_{xx} &= 1, & (x,t) \in (0,1) \times (0,\infty), \\ v_x(0,t) &= 0, & t \in (0,\infty), \\ v_x(1,t) &= 0, & t \in (0,\infty), \\ v(x,0) &= 1 + \cos(2\pi x), & x \in [0,1]. \end{cases}$$

Solving the associated ODE problem coming from the separation of variables and imposing the boundary conditions as before, we get that the possible values of λ (the constant realising $\frac{X''}{X} = \frac{T'}{T} = \lambda$) are given by $\pi^2 n^2$ for $n \in \{0, 1, 2, \dots\}$, and the associated solutions are $X_n(x) = \cos(n\pi x)$. Thus, we are looking for a general solution of the form

$$v(x,t) = \sum_{n \geq 0} T_n(t)\cos(n\pi x).$$

From the initial condition, we directly get that

$$T_0(0) = 1, \quad T_2(0) = 1, \quad T_n(0) = 0, \text{ for all } n \notin \{0, 2\}.$$

On the other hand, imposing that $v_t - v_{xx} = 1$ we get

$$\sum_{n \geq 0} (T_n'(t) + \pi^2 n^2 T_n(t))\cos(n\pi x) = 1.$$

That is,

$$T_0'(t) = 1, \quad T_n'(t) + \pi^2 n^2 T_n(t) = 0, \text{ for all } n \geq 1,$$

And we can solve the various ODE for each T_n:

$(n = 0)$ In this case,

$$T_0'(t) = 1, \quad T_0(0) = 1, \quad \Rightarrow \quad T_0(t) = 1 + t.$$

$(n = 2)$ In this case,

$$T_2'(t) + 4\pi^2 T_2(t) = 0, \quad T_2(0) = 1, \quad \Rightarrow \quad T_2(t) = e^{-4\pi^2 t}.$$

$(n \notin \{0, 2\})$ In this case,

$$T_n'(t) + \pi^2 n^2 T_n(t) = 0, \quad T_n(0) = 0, \quad \Rightarrow \quad T_n(t) = 0.$$

Thus,

$$v(x,t) = 1 + t + e^{-4\pi^2 t}\cos(2\pi x),$$

and, therefore,

$$u(x,t) = v(x,t) + w(x,t) = 1 + t + e^{-4\pi^2 t}\cos(2\pi x) + x\sin(t).$$

(c) Once again, calling v the solution of the homogeneous equation $v_t - v_{xx} = 0$, $v(0,t) = v_x(0,t) = 0$ we have that the ODE obtained by the separation of variables $v(x,t) = X(x)T(t)$ has solutions

- If $\lambda > 0$,
$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$
- If $\lambda = 0$,
$$X(x) = A + Bx.$$
- If $\lambda < 0$,
$$X(x) = A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x).$$

If $\lambda = 0$, from $X(0) = 0$ we get that $A = 0$, and from $X'(\pi) = 0$ we get that $B = 0$, so that only the trivial solution remains.

If $\lambda < 0$, from $X(0) = 0$ we get $A = 0$, and from $X'(\pi) = 0$ we get $B = 0$, so that again, only the trivial solution remains.

Let $\lambda > 0$. From $X(0) = 0$ we get $A = 0$. From $X'(\pi) = 0$ we get $\cos(\sqrt{\lambda}\pi) = 0$. That is,

$$\sqrt{\lambda} = n + \frac{1}{2}, \quad \text{for} \quad n \in \{0, 1, 2, \dots\}.$$

Thus, the set of admissible values for λ is $\lambda_n = \left(n+\frac{1}{2}\right)^2$, and the corresponding solutions are $X_n(x) = \sin\left(\left(n+\frac{1}{2}\right)x\right)$.
We are looking for a general solution of the form

$$u(x,t) = \sum_{n\geq 0} T_n(t) \sin\left(\left(n+\frac{1}{2}\right)x\right)$$

for some functions $T_n(t)$ to be determined. From initial conditions,

$$T_1(0) = 1, \quad T_n(0) = 0, \text{ for all } n \neq 1.$$

Imposing that the equation is fulfilled, we get

$$u(x,t) = \sum_{n\geq 0}\left(T_n'(t) + \left(n+\frac{1}{2}\right)^2 T_n(t)\right)\sin\left(\left(n+\frac{1}{2}\right)x\right) = \sin\left(\frac{9x}{2}\right).$$

Thus, our ODEs are

$(n=1)$ In this case,

$$T_1'(t) + \frac{9}{4}T_1(t) = 0, \quad T_1(0) = 1, \quad \Rightarrow \quad T_1(t) = e^{-\frac{9t}{4}}.$$

$(n=4)$ In this case,

$$T_4'(t) + \frac{81}{4}T_4(t) = 1, \quad T_4(0) = 0, \quad \Rightarrow \quad T_4(t) = \frac{4}{81} - \frac{4}{81}e^{-\frac{81t}{4}}.$$

$(n \notin \{1,4\})$ In this case,

$$T_n'(t) + \left(n+\frac{1}{2}\right)^2 T_n(t) = 0, \quad T_n(0) = 0, \quad \Rightarrow \quad T_n(t) = 0.$$

And our solution is therefore given by

$$u(x,t) = e^{-\frac{9t}{4}}\sin\left(\frac{3x}{2}\right) + \frac{4}{81}\left(1 - e^{-\frac{81t}{4}}\right)\sin\left(\frac{9x}{2}\right).$$

9.2. Conservation of energy Suppose $u(x,t)$ is periodic on $(0,\pi)$ and solves $u_{tt} - u_{xx} = f(x)$, for some periodic function f.

(a) Show that if $f \equiv 0$, then the energy

$$E(t) := \frac{1}{2}\int_0^\pi (u_t(x,t))^2 + (u_x(x,t))^2\,dx,$$

is conserved, i.e. $E(t) = E(0)$ for all $t > 0$.

(b) Inspired by the homogeneous case, find a similar conserved quantity when $f(x) = \sum_{n=1}^M A_n \sin(nx)$.

SOL:

(a) To show that $E(t)$ is constant, we prove $E'(t) = 0$ for all $t > 0$:

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_0^\pi u_t(x,t)\frac{d}{dt}u_t(x,t) + u_x(x,t)\frac{d}{dt}u_x(x,t)\,dx\\ &= \int_0^\pi u_t(x,t)u_{tt}(x,t) + u_x(x,t)u_{xt}(x,t)\,dx\\ &= \int_0^\pi u_t(x,t)u_{xx}(x,t) + u_x(x,t)u_{xt}(x,t)\,dx\\ &= \int_0^\pi -u_{tx}(x,t)u_x(x,t) + u_x(x,t)u_{xt}(x,t)\,dx\\ &= 0,\end{aligned}$$

where in the third line we used $u_{tt} = u_{xx}$, and in the forth line we integrated by parts in x.

(b) Notice that if $F(x)$ is such that $F''(x) = f(x)$, then $w(x,t) := u(x,t) + F(x)$ solves the homogeneous wave equation $w_{tt} - w_{xx} = 0$. Applying the first point to w we get that

$$\begin{aligned}E(t) :&= \frac{1}{2}\int_0^\pi (w_t(x,t))^2 + (w_x(x,t))^2\,dx\\ &= \frac{1}{2}\int_0^\pi (u(x,t)+F(x))_t)^2 + ((u(x,t)+F(x))_x)^2\,dx\\ &= \frac{1}{2}\int_0^\pi (u_t(x,t))^2 + (u_x(x,t)+F'(x))^2\,dx,\end{aligned}$$

is conserved. It is immediate to see that in our case

$$F(x) = \sum_{n=1}^M \frac{-A_n}{n^2}\sin(nx).$$

9.3. Multiple choice Cross the correct answer(s).

(a) Consider the non-homogeneous heat equation

$$\begin{cases} u_t - u_{xx} = p(t)u, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) = u(\pi,t) = 0, & t \in (0,\infty), \\ u(x,0) = f(x), & x \in (0,\pi), \end{cases}$$

where $p(t)$ is a given function of t, and $f(x) = \sum_{n=1}^{M} A_n \sin(nx) \not\equiv 0$. Then, for $k > 1$

X If $p(t) = (k+1)t^k$, $\lim_{t\to+\infty} e^{-t^{k+1}} u(1,t) = 0$.

◯ If $p(t) = (k+1)t^k$, $\lim_{t\to+\infty} u(1,t) = +\infty$.

◯ If $p(t) = \sin(t)$, $\lim_{t\to+\infty} u(1,t) = +\infty$.

X If $p(t) = 1$, $\lim_{t\to+\infty} u(1,t) = A_1 \sin(1)$.

SOL: There are two ways to solve the PDE: the first one is by noticing that [1]

$$(e^{-\int_0^t p(\tau)\,d\tau} u)_t - (e^{-\int_0^t p(\tau)\,d\tau} u)_{xx} = 0.$$

Hence, setting $w(x,t) = e^{-\int_0^t p(\tau)\,d\tau} u(x,t)$, we have (see Exercise 8.1(a)), that

$$w(x,t) = \sum_{n=1}^{M} A_n e^{-n^2 t} \sin(nx),$$

giving

$$u(x,t) = e^{\int_0^t p(\tau)\,d\tau} \sum_{n=1}^{M} A_n e^{-n^2 t} \sin(nx).$$

If this is too tricky for your taste, we can always use the method of separation of variables $u(x,t) = X(x,t)T(x,t)$. We get the two ODEs: $T'(t) - p(t)T(t) - \lambda T(t) = 0$ and $X''(x) - \lambda X(x) = 0$. As usual, the boundary conditions $u(0,t) = u(\pi,t) = 0$ imply that $X(x) = X_n(x) = \sin(nx)$ (again, see Exercise 8.1(a)). On the other side

$$T_n'(t) + (n^2 - p(t))T_n(t) = 0,$$

gives us $T_n(t) = T_n(0) \exp\left\{-n^2 t + \int_0^t p(\tau)\,d\tau\right\}$. The initial datum $u(x,0) = f(x)$ reads

$$\sum_{n=1}^{+\infty} T_n(0) \sin(nx) = \sum_{n=1}^{M} A_n \sin(nx),$$

[1] this nothing else that the method to solve general linear first order ODE: in t for a fixed x the heat equation is of this class. Cf Exercise sheet 1.

giving $T_n(0) = A_n$ for $n = 1, \dots, M$, and $T_n(0) = 0$ for $n > M$. We get the general solution

$$u(x,t) = \sum_{n=1}^{M} A_n \exp\Big\{-n^2 t + \int_0^t p(\tau)\, d\tau\Big\} \sin(nx) = e^{\int_0^t p(\tau)\, d\tau} \sum_{n=1}^{M} A_n e^{-n^2 t} \sin(nx).$$

Having the explicit solution, the correct answers are immediate.

Extra exercises

9.4. Solve the following non-homogeneous problem.

$$\begin{cases} u_t - u_{xx} &= -u, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u(\pi,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= \sin(x), & x \in [0,\pi]. \end{cases}$$

SOL: In this case, the corresponding base of functions is given by $X_n(x) = \sin(nx)$ and $\lambda_n = n^2$ are the admissible values associated. Thus, we are looking for a general solution of the form

$$u(x,t) = \sum_{n \geq 1} T_n(t) \sin(nx).$$

From the initial condition,

$$T_1(0) = 1, \quad T_n(0) = 0, \text{ for all } n \geq 2.$$

On the other hand, imposing that $u_t - u_{xx} + u = 0$,

$$\sum_{n \geq 1} (T_n'(t) + n^2 T_n(t) + T_n(t)) \sin(nx) = 0.$$

That is,

$$T_n'(t) + (n^2+1) T_n(t) = 0, \quad \text{for all } n \geq 1.$$

Solving the corresponding ODEs,

$(n = 1)$ In this case,

$$T_1'(t) + 2T_1(t) = 0, \quad T_1(0) = 1, \quad \Rightarrow \quad T_1(t) = e^{-2t}.$$

$(n \geq 2)$ In this case,

$$T_n'(t) + (n^2+1) T_n(t) = 0, \quad T_n(0) = 0, \quad \Rightarrow \quad T_n(t) = 0.$$

And the general solution is given by

$$u(x,t) = e^{-2t} \sin(x).$$