7.1. (Non)homogeneous wave equation

(a) Let $u = u(x,t)$ be a solution of the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 1, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = 1, & x \in \mathbb{R}, \\ u_t(x,0) = 1, & x \in \mathbb{R}, \end{cases}$$

Compute the explicit solution.

(b) Let $u = u(x,t)$ be a solution of the wave equation

$$\begin{cases} u_{tt} - 2u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = \sin(x), & x \in \mathbb{R}, \end{cases}$$

where $f(x) = x$, if $|x| \leq 2$ and $f(x) = 0$, if $|x| > 2$. Is u smooth? Otherwise, where are the singularities of u? Compute the explicit solution *after* answering these questions.

SOL: Recall the d'Alembert formula for nonhomogenoeus one dimensional wave equations:

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,t)\,d\xi\,d\tau.$$

(a) In this case: $f = g = F = 1$. Hence

$$\begin{aligned} u(x,t) &= 1 + \frac{2ct}{2c} + \frac{1}{2c}\int_0^t 2c(t-\tau)\,d\tau \\ &= 1 + t + \int_0^t (t-\tau)\,d\tau = 1 + t + t^2 - t^2/2 = 1 + t + t^2/2. \end{aligned}$$

(b) Since the functions f and $\sin(x)$ are both odd, and the PDE is homogeneous, we can apply Exercise 6.3, that ensures u to be odd in $x \in \mathbb{R}$ for any fixed $t > 0$. The initial datum has two singularities at $x = \pm 2$. Since we know that singularities travel along characteristics, we have that u is singular along the four lines $x \pm ct = x \pm \sqrt{2}t = \pm 2$.

Apply d'Alembert formula

$$\begin{aligned} u(x,t) &= \frac{f(x+\sqrt{2}t)+f(x-\sqrt{2}t)}{2} + \frac{1}{2\sqrt{2}}\int_{x-\sqrt{2}t}^{x+\sqrt{2}t} \sin(y)\,dy \\ &= \frac{f(x+\sqrt{2}t)+f(x-\sqrt{2}t)}{2} - \frac{\cos(x+\sqrt{2}t)-\cos(x-\sqrt{2}t)}{2\sqrt{2}} \\ &= \frac{f(x+\sqrt{2}t)+f(x-\sqrt{2}t)}{2} + \frac{\sin(x)\sin(\sqrt{2}t)}{\sqrt{2}} \\ &= \frac{\sin(x)\sin(\sqrt{2}t)}{\sqrt{2}} + \begin{cases} 0, & \text{if } |x+\sqrt{2}t|>2 \text{ and } |x-\sqrt{2}t|>2, \\ x, & \text{if } |x+\sqrt{2}t|\le 2 \text{ and } |x-\sqrt{2}t|\le 2 \\ \frac{x\pm\sqrt{2}t}{2}, & \text{if } |x\pm\sqrt{2}t|\le 2 \text{ and } |x\mp\sqrt{2}t|>2. \end{cases} \end{aligned}$$

7.2. Propagation of symmetries from initial data, II Consider the general nonhomogeneous wave equation posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & (x,t) \in \mathbb{R}\times(0,\infty), \\ u(x,0) = f(x), & x\in\mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Take advantage of the uniqueness Theorem (4.4.6) in the notes to show that

(a) if f, g and $F(\cdot,t)$ are odd/even functions, then $u(\cdot,t)$ is itself odd/even.

(b) if f, g and $F(\cdot,t)$ are periodic with same period $T>0$ (i.e. $f(x+T)=f(x)$, $g(x+T)=g(x)$ and $F(x+T,t)=F(x,t)$ for all $x\in\mathbb{R}$ and $t>0$), then $u(\cdot,t)$ is itself periodic with period T.

SOL:

(a) Take $v(x,t) = -u(-x,t)$. Notice that $v_t(x,t) = -u_t(-x,t)$, $v_{tt}(x,t) = -u_{tt}(-x,t)$ on the one hand, and $v_x(x,t) = u_x(-x,t)$, and $v_{xx}(x,t) = -u_{xx}(-x,t)$. Thus,

$$v_{tt}(x,t) - c^2 v_{xx}(x,t) = -u_{tt}(-x,t) + c^2 u_{xx}(-x,t) = -F(-x,t) = F(x,t),$$

where in the last equality we are using that F is spatially odd. Similarly,

$$v(x,0) = -u(-x,0) = -f(-x) = f(x), \quad v_t(x,0) = -u_t(-x,0) = -g(-x) = g(x).$$

Thus, v satisfies

$$\begin{cases} v_{tt} - c^2 v_{xx} = F(x,t), & (x,t)\in\mathbb{R}\times(0,\infty), \\ v(x,0) = f(x), & x\in\mathbb{R}, \\ v_t(x,0) = g(x), & x\in\mathbb{R}, \end{cases}$$

that is, v and u satisfy the same problem, (1). Since (1) has a unique solution, we must have $v(x,t) = u(x,t)$ for all $x \in \mathbb{R}$, $t \geq 0$; that is, $-u(-x,t) = u(x,t)$, u is spatially odd. If f, g, and F are spatially even (even with respect to x), then u is also spatially even. (That is, $f(-x) = f(x)$, $g(-x) = g(x)$ and $F(-x,t) = F(x,t)$ imply $u(-x,t) = u(x,t)$.)

The solution is the same as above, taking $v(x,t) = u(-x,t)$ instead.

(b) If f, g, and F are L-periodic, then u is also L-periodic. (That is, $f(x) = f(x+L)$, $g(x) = g(x+L)$ and $F(x,t) = F(x+L,t)$, imply $u(x,t) = u(x+L,t)$.)

The solution is the same as above, taking $v(x,t) = u(x+L,t)$ instead.

7.3. Wave equation on a ring Let $u : [0,1] \times [0,\infty) \to \mathbb{R}$ be a solution of the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in [0,1] \times (0,\infty), \\ u(x,0) = x - x^2, & x \in [0,1], \\ u_t(x,0) = 0, & x \in [0,1]\,, \\ u(0,t) = u(1,t), & t \in (0,\infty)\,, \\ u_x(0,t) = u_x(1,t), & t \in (0,\infty)\,. \end{cases}$$

Compute $u(1/2, 2022)$.

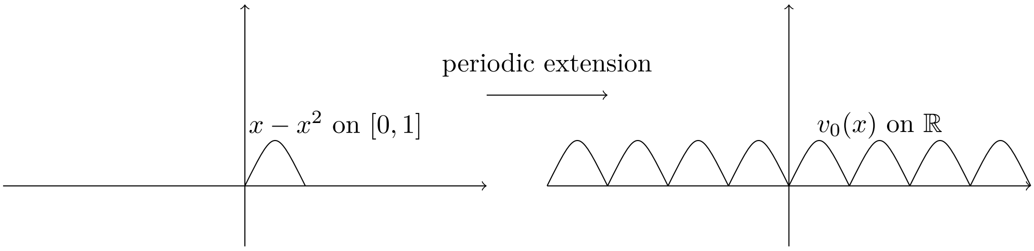
SOL: A general consideration: Up to now, the only tool we have to explicitly compute u is the d'Alembert formula, which can be applied only when the PDE takes place on the whole real line $\mathbb{R}$. In general one can be interested in solving a wave equation taking place in a smaller domain $D \subset \mathbb{R}$. The trick is the following: we *extend* the boundary data of the PDE on the whole line producing an auxiliary problem that we can solve with d'Alembert. Then, we restrict the computed solution on D, and we check taking advantage of Exercise 6.3/7.2 that the restriction solves the original PDE. The way the extension should be operated is suggested by the additional boundary conditions of the problem. For instance, Exercise 6.6 was solved in this way. This exercise is another example of this general method.

In this particular case we want to solve the wave equation on $[0,1]$, and we search for a PDE on the whole line of the form

$$\begin{cases} v_{tt} - v_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) = v_0(x), & x \in \mathbb{R}, \\ v_t(x,0) = 0, & x \in \mathbb{R}\,, \end{cases} \tag{2}$$

such that setting $u(x,t) := v(x,t)$ for $(x,t) \in [0,1] \times (0,\infty)$, u solves the original problem. The question now is how v_0 should be defined. Of course $v_0(x,t) = x - x^2$

for $x \in [0,1]$ since we want that $v\mid_{[0,1]}= u$. The additional boundary conditions $u(0,t) = u(1,t)$ and $u_x(0,t) = u_x(1,t)$ are imposing the solution to be *periodic*. The natural way to define v_0 is therefore by periodicity: $v_0(x) = (x - \lfloor x \rfloor) - (x - \lfloor x \rfloor)^2$, where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : 0 \le x - n < 1\}$. This is just a fancy notation for the natural periodic extension showed in the picture below



We know that the solution exists and is periodic (with period 1) as shown in *(c)* of the previous exercise. Let u be the restriction of v in the domain $[0,1] \times [0,\infty)$. Clearly u satisfies the wave equation in the domain and $u(x,0) = v(x,0) = x - x^2$ and $u_t(x,0) = v_t(x,0) = 0$. Moreover, thanks to the periodicity of v, we have

$$u(0,t) = v(0,t) = v(1,t) = u(1,t)\,,$$
$$u_x(0,t) = v_x(0,t) = v_x(1,t) = u_x(1,t)\,.$$

Thus the function u satisfies the PDE given in the statement. With a similar argument one can also prove that this u is the unique solution (the idea is to define v as the extension of u and show that it satisfies (2), so that we can invoke the uniqueness for the classical wave equation).

To compute the value of $u(1/2, 2022)$ we exploit the d'Alembert formula for v:

$$u(1/2, 2022) = v(1/2, 2022) = \frac{1}{2}(v_0(1/2 - 2022) + v_0(1/2 + 2022)) = v_0(1/2) = 1/4.$$

7.4. Multiple choice Cross the correct answer(s).

(a) Let u be solution of the homogeneous wave equation

$$\begin{cases} u_{tt} - 9u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Let h be a smooth function, and u_h be the solution of the above PDE with perturbed initial condition $u_h(x,0) = f(x)$ and $(u_h)_t(x,0) = g(x) + h(x)$. Then, $u(1,2) = u_h(1,2)$

- ◯ whenever h has compact support in $[-5,7]$
- ◯ only when h constantly equal to zero
- X whenever $\int_{-5}^{7} h(x)\,dx = 0$
- X whenever h us equal to zero in $[-5,7]$

SOL: We know that in general the value of $u(x_0,t_0)$ is uniquely determined by the values of u in the *triangle of influence*. In the particular case of the homogeneous wave equation, we can actually say more: looking at the d'Alembert formula, we see that $u(x_0,t_0)$ depends uniquely of the value of f on the two points $x_0 - ct_0$ and $x_0 + ct_0$, and the integral of g on the interval $(x_0 - ct_0, x_0 + ct_0)$.

(b) Same question as (a), but when we perturb $u_h(x,0) = f(x) + h(x)$, $(u_h)_t(x,0) = g(x)$.

- ◯ only when h constantly equal to zero
- ◯ always for h small enough
- X when $h(x) = \sin(\pi(x+1))$
- X whenever $h(-5) = h(7) = 0$

SOL: Same reason of question (a).

(c) Let u be solution of the homogeneous wave equation

$$\begin{cases} u_{tt} - u_{xx} = F(x), & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Suppose that F, f and g are trigonometric polynomials as in Exercise 6.2 (b) with $\int_0^{2\pi} g\,dx = 0$. Then, u is

- ◯ never
- X always for $F \equiv 0$
- ◯ always
- ◯ never unless $F \neq f$

2π-periodic *in time*[1], that is $u(x, t + 2\pi) = u(x,t)$ for all $(x,t) \in \mathbb{R} \times (0, +\infty)$.

SOL: Observe if

$$g(x) = \sum_{n=0}^{N} b_n \cos(nx),$$

then $b_0 = \frac{1}{2\pi} \int_0^{2\pi} g(x)\,dx = 0$. Looking at the general solution of Exercise 6.2 (b), we infer that when $F \equiv 0$, i.e. the PDE is homogeneous, then for a fixed $x \in \mathbb{R}$, u is periodic in t. Take $F \equiv a \in \mathbb{R} \setminus \{0\}$ as a counter example for all the other points.

[1] Be careful, this is not the same as being periodic in the x variable, as in Exercise 7.2.

Extra exercises

7.5. Strange wave equation Show that the following partial differential equation admits a solution

$$\begin{cases} u_{tt} - u_{xx} = \frac{u_t^2 - u_x^2}{2u}, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = x^4, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Hint: Consider the function $v(x,t) = \sqrt{u(x,t)}$. What equation does it satisfy?

SOL: We assume that u is a solution of the PDE and we compute the derivatives of $v(x,t) := \sqrt{u(x,t)}$. We have

$$\begin{aligned} v_t &= \frac{1}{2} u^{-\frac{1}{2}} u_t \,, \\ v_{tt} &= \frac{-1}{4} u^{-\frac{3}{2}} u_t^2 + \frac{1}{2} u^{-\frac{1}{2}} u_{tt} \,, \\ v_x &= \frac{1}{2} u^{-\frac{1}{2}} u_x \,, \\ v_{xx} &= \frac{-1}{4} u^{-\frac{3}{2}} u_x^2 + \frac{1}{2} u^{-\frac{1}{2}} u_{xx} \,. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} v_{tt} - v_{xx} &= \frac{-1}{4} u^{-\frac{3}{2}} u_t^2 + \frac{1}{2} u^{-\frac{1}{2}} u_{tt} + \frac{1}{4} u^{-\frac{3}{2}} u_x^2 - \frac{1}{2} u^{-\frac{1}{2}} u_{xx} \\ &= \frac{1}{2} u^{-\frac{1}{2}} (u_{tt} - u_{xx}) - \frac{1}{4} u^{-\frac{3}{2}} (u_t^2 - u_x^2) \\ &= \frac{1}{4} u^{-\frac{3}{2}} (u_t^2 - u_x^2) - \frac{1}{4} u^{-\frac{3}{2}} (u_t^2 - u_x^2) = 0 \,. \end{aligned}$$

Hence we have proven that v satisfies

$$\begin{cases} v_{tt} - v_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) = x^2, & x \in \mathbb{R}, \\ v_t(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{3}$$

Up to now, we have just noticed that if u is a solution of the original PDE then $\sqrt{u}$ solves the wave equation; but we have never shown the existence of u. In order to do so, let $v(x,t)$ be the solution of (3) and define $u := v^2$ (our computations justify this choice). We want to check that u is a solution of the original PDE.

Thanks to the d'Alembert's formula we know

$$v(x,t) = x^2 + t^2$$

and therefore

$$u(x,t) = x^4 + t^4 + 2x^2t^2$$

is our candidate solution. Checking the initial conditions $u(x,0) = x^4$ and $u_t(x,0) = 0$ is immediate. Hence, we just have to check whether u solves the equation. We have

$$\begin{aligned} u_{tt} - u_{xx} &= 12t^2 + 4x^2 - (12x^2 + 4t^2) = 8(t^2 - x^2)\,, \\ u_t &= 4t^3 + 4x^2t = 4t(x^2 + t^2) \implies u_t^2 = 16t^2u\,, \\ u_x &= 4x^3 + 4tx^2 = 4x(x^2 + t^2) \implies u_x^2 = 16x^2u\,. \end{aligned}$$

Therefore we get

$$u_{tt} - u_{xx} = 8(t^2 - x^2) = \frac{16t^2u - 16x^2u}{2u} = \frac{u_t^2 - u_x^2}{2u}$$

which is exactly the desired partial differential equation. Thus we have shown that $u(x,t) = x^4 + t^4 + 2x^2t^2$ is a solution.