6.1. Wave equation Consider the homogeneous one dimensional wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = \sin(x) + 2\cos(x), & x \in \mathbb{R}, \\ u_t(x,0) = 1, & x \in \mathbb{R}. \end{cases}$$

Compute the explicit solution u.

SOL: We apply d'Alembert formula

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy,$$

where here $c = 2$, $f(x) = \sin(x) + 2\cos(x)$ and $g(x) = 1$. We get

$$\begin{aligned} u(x,t) &= \frac{\sin(x+2t) + 2\cos(x+2t) + \sin(x-2t) + 2\cos(x-2t)}{2} + \frac{1}{4}(x + 2t - x + 2t) \\ &= \sin(x)\cos(2t) + 2\cos(x)\cos(2t) + t, \end{aligned}$$

where we used $\sin(a+b) + \sin(a-b) = 2\sin(a)\cos(b)$ and $\cos(a+b) + \cos(a-b) = 2\cos(a)\cos(b)$.

6.2. Wave equation's anatomy Consider the general homogeneous one dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

(a) Identify the backward and forward waves F and G. Give the necessary and sufficient condition on f and g to have $F = G$, or $F = -G$.

(b) Suppose that $f(x)$ and $g(x)$ are trigonometric polynomials of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{N} a_n \cos(nx), \\ g(x) &= \sum_{n=0}^{M} b_n \cos(nx), \end{aligned}$$

where $\{a_n\}_{n=0}^N, \{b_n\}_{n=0}^M \subset \mathbb{R}$, and $N, M \geq 0$. Find a general solution u.

SOL:

(a) Recall that the solution of the wave equation can always be decomposed as

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are respectively the backward and forward wave. Looking at d'Alembert formula, setting $\bar{g}$ equal to the the primitive of g we have that

$$\begin{aligned} u(x,t) &= \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy \\ &= \frac{1}{2c}(cf(x+ct)+\bar{g}(x+ct)) + \frac{1}{2c}(cf(x-ct)-\bar{g}(x-ct)), \end{aligned}$$

hence $F(\xi) = \frac{1}{2c}(cf(\xi)+\bar{g}(\xi))$ and $G(\xi) = \frac{1}{2c}(cf(\xi)-\bar{g}(\xi))$. If $F = G$ then

$$F(\xi) = \frac{1}{2c}(cf(\xi)+\bar{g}(\xi)) = \frac{1}{2c}(cf(\xi)-\bar{g}(\xi)) = G(\xi),$$

equivalently

$$\bar{g} = -\bar{g},$$

which implies that $\bar{g} = 0$, and then $g = 0$. Similarly, $G = -F$ if and only if $f = 0$.

(b) Applying the d'Alembert formula for the given trigonometric polynomials, we get that

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{N} a_n \frac{\cos(n(x+ct))+\cos(n(x-ct))}{2} + \frac{1}{2c}\sum_{n=0}^{M} b_n \int_{x-ct}^{x+ct}\cos(ny)\,dy \\ &= \sum_{n=0}^{N} a_n\cos(nct)\cos(nx) + \sum_{n=1}^{M}\frac{b_n}{2cn}(\sin(n(x+ct))-\sin(n(x-ct))) \\ &+ \frac{b_0}{2c}\int_{x-ct}^{x+ct} 1\,dy \\ &= \sum_{n=0}^{N} a_n\cos(nct)\cos(nx) + \sum_{n=1}^{M}\frac{b_n}{cn}\sin(nct)\cos(nx) + b_0 t, \end{aligned}$$

where we used $\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$, $\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$.

6.3. Propagation of symmetries from initial data

Consider the general wave equation posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in \mathbb{R}\times(0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

(a) Suppose that both f and g are odd functions (that is, $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$). Show that the solution u is also an odd function in x, for each time $t > 0$ (that is, $u(-x,t) = -u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$).

(b) Suppose now that both f and g are even (that is $f(x) = f(-x)$ and $g(x) = g(-x)$ for all $x \in \mathbb{R}$), and 2π-periodic functions. Using Exercise 6.2 (b), justify formally why $u(\cdot,t)$ has to be an even function.[1]

(c) Find an explicit example in which f is even, g is odd, and the solution is for $t > 0$ neither even nor odd.

SOL:

(a) By d'Alembert formula

$$\begin{aligned}
-u(x,t) &= -\frac{f(x+ct)+f(x-ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy \\
&= \frac{f(-(x+ct))+f(-(x-ct))}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(-y)\,dy \\
&= \frac{f(-x-ct)+f(-x+ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} g(z)\,dz \\
&= u(-x,t),
\end{aligned}$$

where in the second line we used that f and g are odd, and in the last the change of variable in the integral $z = -y$.

(b) Since f and g are periodic, we can develop as Fourier infinite series

$$f(x) = \sum_{n=0}^{+\infty} a_n \cos(nx) + b_n \sin(nx),$$

and

$$g(x) = \sum_{n=0}^{+\infty} A_n \cos(nx) + B_n \sin(nx).$$

Moreover, since both are even, we know that b_n and B_n must be equal to zero (recall that the Fourier serie of an even function has only the cosinus terms). By Exercise 6.2 (b), we know that for every fixed time t, u has the Fourier series in the form

$$u(x,t) = \sum_{n=0}^{+\infty} c_n(t)\cos(nx),$$

[1] One can actually argue as in point (a) without needing the trigonometric expansions of exercise 6.2 (b). The purpose of this point is to see that the Fourier expansion gives another tool to investigate the symmetries of u.

where $c_n(t) = a_n \cos(nct) + \frac{A_n}{cn}\sin(nct)$. It follows that u must also be even, since it is composed uniquely by cosinus terms.

(c) There are many examples. For example, $f(x) = \cos(x)$ is even, $g(x) = x$ is odd and $u(x,t) = \cos(x)\cos(ct) + tx$ is not even nor odd for $t > 0$.

6.4. Multiple choice Cross the correct answer(s).

(a) The second order linear PDE given by

$$u_x + x^2 u_{xx} + 2x\sin(y)u_{xy} - \cos^2(y)u_{yy} + e^x = 0,$$

is

- X hyperbolic if $x \neq 0$
- ○ everywhere hyperbolic
- ○ parabolic in $\{y = k\frac{\pi}{2} : k \in \mathbb{Z}\}$
- X parabolic in $x = 0$

SOL: we compute $\delta = b^2 - ac = x^2\sin^2(y) + x^2\cos^2(y) = x^2 > 0$ unless $x = 0$.

(b) Consider u solution of the one dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = \cos(t), & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = x, & x \in \mathbb{R}, \\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Then,

- X for all $x \in \mathbb{R}$ one has that $\lim_{t\to+\infty} \frac{u(x,t)}{t} = 0$
- ○ u is odd in x
- X $u(x,t)$ is periodic in t
- ○ u is periodic in x
- ○ u is odd in t

SOL: Applying d'Alembert:

$$\begin{aligned} u(x,t) &= \frac{x+t+x-t}{2} + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \cos(\tau)\, d\xi\, d\tau \\ &= x + \int_0^t (t-\tau)\cos(\tau)\, d\tau \\ &= x + 1 - \cos(t). \end{aligned}$$

We can see that for a fixed x, $u(x,t)/t$ goes to zero since $|\cos(t)| \leq 1$.

(c) Consider u solution of the one dimensional wave equation

$$\begin{cases} u_{tt} - \pi^2 u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0, +\infty), \\ u(x,0) = x^2, & x \in \mathbb{R}, \\ u_t(x,0) = -\sin(x), & x \in \mathbb{R}. \end{cases}$$

The value of u at $(x,t) = (\pi, 2)$ is equal to

X $5\pi^2$

○ $3\pi^2$

○ 0

○ 2π

SOL: By the d'Alembert formula

$$\begin{aligned} u(\pi, 2) &= \frac{(\pi - 2\pi)^2 + (\pi + 2\pi)^2}{2} - \frac{1}{2\pi} \int_{\pi-2\pi}^{\pi+2\pi} \sin(y)\, dy \\ &= 5\pi^2 - \frac{1}{2\pi} \int_{-\pi}^{3\pi} \sin(y)\, dy \\ &= 5\pi^2. \end{aligned}$$

Extra exercises

6.5. Time reversible

Consider the Cauchy problem posed for $-\infty < x < \infty$ and $t > 0$,

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0, & (x,t) \in \mathbb{R} \times (0, \infty), \\ u(x,0) &= f(x), & x \in \mathbb{R}, \\ u_t(x,0) &= g(x), & x \in \mathbb{R}. \end{cases}$$

Let $\tilde{u}(x,t) := u(x,-t)$. Show that $\tilde{u}(x,t)$ solves the Cauchy problem posed for $-\infty < x < \infty$ and $t < 0$,

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} &= 0, & (x,t) \in \mathbb{R} \times (-\infty, 0), \\ \tilde{u}(x,0) &= f(x), & x \in \mathbb{R}, \\ \tilde{u}_t(x,0) &= -g(x), & x \in \mathbb{R}. \end{cases}$$

That is, we are showing that the wave equation is reversible in time. If a function solves a wave equation, the same function with time reversed also solves a the wave equation with the same initial condition and opposite initial velocity.

SOL: We just need to check the properties one by one.

First notice that
$$\tilde{u}(x,0) = u(x,0) = f(x)$$
and
$$\tilde{u}_t(x,0) = \frac{d}{dt}(u(x,-t))\Big|_{t=0} = -u_t(x,0) = -g(x),$$
so that the initial conditions hold. Now, since u is defined for $(x,t) \in \mathbb{R} \times (0,\infty)$, then $\tilde{u}$ is defined for $(x,t) \in \mathbb{R} \times (-\infty, 0)$. Finally, notice that
$$\tilde{u}_{tt}(x,t) = (u(x,-t))_{tt} = -(u_t(x,-t))_t = u_{tt}(x,-t)$$
and similarly
$$\tilde{u}_{xx}(x,t) = (u(x,-t))_{xx} = u_{xx}(x,-t).$$
Therefore,
$$\tilde{u}_{tt}(x,t) - c^2\tilde{u}_{xx}(x,t) = u_{tt}(x,-t) - c^2 u_{xx}(x,-t) = 0$$
where we are using the original equation, $u_{tt} - c^2 u_{xx} = 0$.

6.6. Zero boundary condition

Use the previous exercise to solve the following Cauchy problem posed for $x > 0$ and $t > 0$, with zero boundary condition at $x = 0$,
$$\begin{cases} u_{tt} - u_{xx} &= 0, & (x,t) \in (0,\infty) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= x^2, & x \in (0,\infty), \\ u_t(x,0) &= 0, & x \in (0,\infty). \end{cases}$$

SOL:

Notice that our problem is now posed on the half-line, $x > 0$, with zero boundary condition at $x = 0$ for all times $t > 0$. The initial value is x^2, which when evaluated at 0 is consistent with the boundary condition.

The previous exercise tells us that solutions starting from an odd initial value, remain odd at all times. In particular, we know that continuous odd functions must be 0 at 0: that is, if $f(-x) = -f(x)$ for all x, then for $x = 0$ we get $f(0) = -f(0)$ which means $f(0) = 0$. Then, the zero boundary condition at all times posed at $x = 0$ will hold if the solution is odd at all times. Thus, thanks to the previous exercise, it will be enough to solve the problem in the whole line
$$\begin{cases} u_{tt} - u_{xx} &= 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= f(x), & x \in \mathbb{R}, \\ u_t(x,0) &= 0, & x \in \mathbb{R}, \end{cases}$$

where $f(x) = x^2$ if $x > 0$ and $f(x) = -x^2$ if $x \leq 0$, is the odd extension of x^2 to the whole $\mathbb{R}$. Alternatively, we can write $f(x) = x|x|$. Thus, our solution, by d'Alembert formula (which works only if the domain is the whole real line $\mathbb{R}$), is given by

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} = \frac{(x+t)|x+t| + (x-t)|x-t|}{2}.$$

More explicitly, we can separate in three different cases:

- If $x \geq t$, then $x + t \geq 0$ and $x - t \geq 0$, so that
$$u(x,t) = \frac{(x+t)^2 + (x-t)^2}{2} = x^2 + t^2.$$
- If $-t < x < t$, then $x - t < 0$ and $x + t > 0$, so that
$$u(x,t) = \frac{(x+t)^2 - (x-t)^2}{2} = 2xt.$$
- If $x \leq -t$, then $x + t \leq 0$ and $x - t \leq 0$, so that
$$u(x,t) = -\frac{(x+t)^2 + (x-t)^2}{2} = -x^2 - t^2.$$

By construction, it is clear that the function u restricted to $\{x \geq 0\}$ solves the PDE on the half line as wished.