2.1. Method of characteristics I Solve the following equations using the method of characteristics.

(a) $u_x + u_y = 1$, with $u(x, 0) = f(x)$.

(b) $xu_x + (x + y)u_y = 1$, with $u(1, y) = y^2$.

(c) $u_x - 2xyu_y = 0$, with $u(0, y) = y$.

(d) $yu_x - xu_y = 0$, with $u(x, 0) = g(x^2)$ for all $x > 0$.

SOL:

(a) Step by step: this PDE is on the form $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$, with $a(x, y, u) = b(x, y, u) = c(x, y, u) = 1$. The initial curve is $\Gamma(s) = (s, 0, f(s))$. The associated ODE system for the characteristic curves $x(t, s)$, $y(t, s)$, $\tilde{u}(t, s)$ is given by

$$\begin{cases} \frac{dx(t,s)}{dt} = a(x(t, s), y(t, s), \tilde{u}(t, s)) = 1, & x(0, s) = s, \\ \frac{dy(t,s)}{dt} = b(x(t, s), y(t, s), \tilde{u}(t, s)) = 1, & y(0, s) = 0, \\ \frac{d\tilde{u}(t,s)}{dt} = c(x(t, s), y(t, s), \tilde{u}(t, s)) = 1, & \tilde{u}(0, s) = f(s). \end{cases}$$

We can solve each ODE separately, obtaining

$$\begin{cases} x(t, s) = t + s, \\ y(t, s) = t, \\ \tilde{u}(t, s) = t + f(s). \end{cases}$$

Inverting the map $(t, s) \mapsto (x(t, s), y(t, s))$ is elementary: $t = y$ and $s = x - t = x - y$. Hence $u(x, t) = \tilde{u}(x(t, s), y(t, s)) = t + f(s) = y + f(x - y)$.

(b) Step by step: this PDE is on the form $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$, with $a(x, y, u) = x$, $b(x, y, u) = (x + y)$ and $c(x, y, u) = 1$. The initial curve is $\Gamma(s) = (1, s, s^2)$. The associated ODE system for the characteristic curves $x(t, s)$, $y(t, s)$, $\tilde{u}(t, s)$ is given by

$$\begin{cases} \frac{dx(t,s)}{dt} = a(x(t, s), y(t, s), \tilde{u}(t, s)) = x(t, s), & x(0, s) = 1, \\ \frac{dy(t,s)}{dt} = b(x(t, s), y(t, s), \tilde{u}(t, s)) = x(t, s) + y(t, s), & y(0, s) = s, \\ \frac{d\tilde{u}(t,s)}{dt} = c(x(t, s), y(t, s), \tilde{u}(t, s)) = 1, & \tilde{u}(0, s) = s^2. \end{cases}$$

From the first equation we get $x(t, s) = e^t$. Plugging this in the second equation, we get $\frac{dy}{dt} - y = x = e^t$. Multiplying by e^{-t} we have that[1]

$$\frac{d}{dt}(e^{-t}y) = 1.$$

[1] If you don't remember how to solve linear ODEs, look back to the solutions of Exercise 1.2

Intergrating both sides we get $y(t,s) = (s+t)e^t$. Finally, $\tilde{u}(t,s) = t + s^2$. Inverting the map $(t,s) \mapsto (x(t,s), y(t,s))$ gives $t = \ln(x)$ and $s = \frac{y}{x} - \ln(x)$. Hence $u(x,t) = \tilde{u}(x(t,s), y(t,s)) = t + s^2 = \ln(x) + \left(\frac{y}{x} - \ln(x)\right)^2$.

(c) Step by step: this PDE is on the form $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$, with $a(x,y,u) = 1$, $b(x,y,u) = -2xy$ and $c(x,y,u) = 0$. The initial curve is $\Gamma(s) = (0,s,s)$. The associated ODE system for the characteristic curves $x(t,s)$, $y(t,s)$, $\tilde{u}(t,s)$ is given by

$$\begin{cases} \frac{dx(t,s)}{dt} = a(x(t,s), y(t,s), \tilde{u}(t,s)) = 1, & x(0,s) = 0, \\ \frac{dy(t,s)}{dt} = b(x(t,s), y(t,s), \tilde{u}(t,s)) = -2x(t,s)y(t,s), & y(0,s) = s, \\ \frac{d\tilde{u}(t,s)}{dt} = c(x(t,s), y(t,s), \tilde{u}(t,s)) = 0, & \tilde{u}(0,s) = s. \end{cases}$$

From the first equation we get $x(t,s) = t$. Plugging this in the second equation, we have $\frac{dy}{dt} = -2ty$. Dividing by y and integrating we get

$$\ln(y) - \ln(s) = -t^2,$$

so that $y(t,s) = se^{-t^2}$. Clearly, $\tilde{u}(t,s) = s$. Inverting the map $(t,s) \mapsto (x(t,s), y(t,s))$ gives $t = x$ and $s = ye^{x^2}$. Hence $u(x,t) = \tilde{u}(x(t,s), y(t,s)) = s = ye^{x^2}$.

(d) Step by step: this PDE is on the form $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$, with $a(x,y,u) = y$, $b(x,y,u) = -x$ and $c(x,y,u) = 0$. The initial curve is $\Gamma(s) = (s, 0, g(s^2))$. The associated ODE system for the characteristic curves $x(t,s)$, $y(t,s)$, $\tilde{u}(t,s)$ is given by

$$\begin{cases} \frac{dx(t,s)}{dt} = a(x(t,s), y(t,s), \tilde{u}(t,s)) = y(t,s), & x(0,s) = s, \\ \frac{dy(t,s)}{dt} = b(x(t,s), y(t,s), \tilde{u}(t,s)) = -x(t,s), & y(0,s) = 0, \\ \frac{d\tilde{u}(t,s)}{dt} = c(x(t,s), y(t,s), \tilde{u}(t,s)) = 0, & \tilde{u}(0,s) = g(s^2). \end{cases}$$

Since the ODE for x depends to y and viceversa, here we need to be a bit more careful. The trick is to differentiate again in t: from $\frac{dx}{dt} = y$ we get $\frac{d^2x}{dt^2} = \frac{dy}{dt} = -x$, by the second equation. By Exercise 1.2 (e), we know that the general solution of $\frac{d^2x}{dt^2} + x = 0$ is

$$x(t,s) = A\cos(t) + B\sin(t).$$

Evaluating at $t = 0$, we get $x(0,s) = A = s$. Doing the same for y, from $\frac{d^2y}{dt^2} = -\frac{dx}{dt} = -y$, we obtain $y(t,s) = C\cos(t) + D\sin(t)$. Evaluating in $t = 0$, we get $y(0,s) = C = 0$. Therefore, we are left to determinate the constants B and D. Plugging this back to the initial ODEs we have

$$\frac{dx}{dt} = \frac{d}{dt}(s\cos(t) + B\sin(t)) = -s\sin(t) + B\cos(t) = y = D\sin(t),$$

and

$$\frac{dy}{dt} = \frac{d}{dt} D\sin(t) = D\cos(t) = -x = s\cos(t) + B\sin(t).$$

This implies $D = -s$ and $B = 0$, hence

$$\begin{cases} x(t,s) = s\cos(t), \\ y(t,s) = -s\sin(t). \end{cases}$$

Finally, $\tilde{u}(t,s) = g(s^2)$. Notice that $s^2 = x^2 + y^2$, hence $u(x,y) = g(x^2 + y^2)$.

2.2. Method of characteristics II Consider the PDE

$$u_x + (x+y)u_y = 1.$$

Solve the general system of ODEs associated to this PDE. Then, for each initial data listed below, find an explicit solution via the Method of Characteristics if possible. If it is not possible, explain why.

(a) $u(0,y) = 1 - y$.

(b) $u(x, -1-x) = e^x$, $x \in \mathbb{R}$.

SOL: The PDE is of the form $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$ with $a(x,y,u) = 1$, $b(x,y,u) = x + y$ and $c(x,y,u) = 1$. The initial curve is not defined so far, so we just set it generically $\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s))$. Then, the associated system of ODEs is

$$\begin{cases} \frac{dx(t,s)}{dt} = a(x(t,s), y(t,s), \tilde{u}(t,s)) = 1, & x(0,s) = x_0(s), \\ \frac{dy(t,s)}{dt} = b(x(t,s), y(t,s), \tilde{u}(t,s)) = x(t,s) + y(t,s), & y(0,s) = y_0(s), \\ \frac{d\tilde{u}(t,s)}{dt} = c(x(t,s), y(t,s), \tilde{u}(t,s)) = 1, & \tilde{u}(0,s) = \tilde{u}_0(s). \end{cases}$$

We get immediately that $x(t,s) = t + x_0(s)$ and $\tilde{u}(t,s) = t + \tilde{u}_0(s)$. Plugging $x = t + x_0$ in the equation for y we get

$$\frac{dy}{dt} - y = t + x_0(s).$$

Multiplying by e^{-t} on both sides we have that

$$e^{-t}\frac{dy}{dt} - e^{-t}y = \frac{d}{dt}(e^{-t}y) = te^{-t} + e^{-t}x_0(s).$$

Integrating both sides we get

$$\begin{aligned} e^{-t}y(t,s) - y_0(s) &= \int_0^t \tau e^{-\tau} + e^{-\tau}x_0(s)\,d\tau = (1 - e^{-t})x_0(s) + \int_0^t \tau e^{-\tau}\,d\tau \\ &= (1 - e^{-t})x_0(s) - te^{-t} - e^{-t} + 1, \end{aligned}$$

where we solved the second integral by parts. We get

$$\begin{cases} x(t,s) = t + x_0(s), \\ y(t,s) = (e^t - 1)(x_0(s) + 1) - t + e^t y_0(t). \end{cases}$$

(a) In this case $\Gamma(s) = (0, s, 1-s)$, obtaining

$$\begin{cases} x(t,s) = t, \\ y(t,s) = e^t - 1 - t + se^t, \\ \tilde{u}(s,t) = t + 1 - s. \end{cases}$$

Notice that $s = e^{-t}(y - e^t + 1 + t) = e^{-x}(y - e^x + 1 + x) = e^{-x}y - 1 + e^{-x} + xe^{-x}$, giving

$$\tilde{u}(x,y) = t + 1 - s = x - e^{-x}(1 + x + y) + 2.$$

(b) In this case $\Gamma(s) = (s, -1-s, e^s)$, obtaining

$$\begin{cases} x(t,s) = t + s, \\ y(t,s) = (e^t - 1)(s+1) - t - e^t(s+1) = -s - 1 - t, \\ \tilde{u}(s,t) = t + e^s. \end{cases}$$

We have to invert the map $(t,s) \mapsto (x(t,s), y(t,s)) = (t+s, -(t+s) - 1)$. But this is not possible because it is not a bijection! The method of characteristic fails in this case. A geometric reason: observe that the characterisitcs $t \mapsto (x(t,s), y(t,s)) = (t+s, -(t+s)-1)$ never leave the initial curve $\Gamma(s)$.

2.3. Multiple choice Cross the correct answer(s).

(a) The expression $f(u_{xxx}) = u_z + 5$ describes a quasilinear PDE of order 3 if

- ○ f is linear
- ○ f is constant
- X f is invertible
- ○ f is a polynomial

SOL: If f is invertible, then $u_{xxx} = f^{-1}(u_z + 5)$ is quasilinear. This is not true for f equal to a constant because we obtain a linear PDE $u_z + 5 = \text{const}$. The same holds for the other options because for example $f \equiv 0$ is constant and polynomial at the same time.

(b) The Hessian of a C^2-function $u : \mathbb{R}^n \to \mathbb{R}$ is the $n \times n$ symmetric matrix D^2u, whose coefficients are $(D^2u)_{ij} = u_{x_i x_j}$, $i, j \in \{1, \dots, n\}$. For $n \geq 2$ the *Monge-Ampère equations* are the PDEs in the form: $\det(D^2u) = f$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a given smooth function. These PDEs are

X fully nonlinear
◯ quasilinear
◯ linear homogeneous if $f \equiv 0$
◯ linear inhomogeneous if $f \not\equiv 0$
X of second order
◯ of third order

SOL: It suffices to look at the PDE when $n = 2$ to convince yourself that $\det(D^2u) = u_{xx}u_{yy} - (u_{xy})^2 = f$ is fully nonlinear and of second order.

(c) Let $\Omega \subset \mathbb{R}^2$. Given a function $H : \Omega \to \mathbb{R}$, to find a function $u : \Omega \to \mathbb{R}$ whose surface graph $\Sigma = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$ has *mean curvature* equal to $H(x, y)$ at each point $(x, y, u(x, y)) \in \Sigma$, one has to solve the *prescribed mean curvature equation*:

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = H.$$

Here $\nabla u = (u_x, u_y)$ and $|\nabla u|^2 = (u_x)^2 + (u_y)^2$. This PDE is

◯ fully nonlinear
X quasilinear
◯ linear homogeneous if $H \equiv 0$
◯ linear inhomogeneous if $H \not\equiv 0$
X of second order
◯ of third order

SOL: Develop the equation:

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \frac{\Delta u}{\sqrt{1+|\nabla u|^2}} - \frac{u_x(u_xu_{xx} + u_yu_{xy}) + u_y(u_xu_{xy} + u_yu_{yy})}{(1+|\nabla u|^2)^{3/2}}.$$

(d) Consider the PDE $yu_x - x^2u_y = 0$ coupled with the boundary condition $u(x, y) = 2$ on $\{(x, y) : x^3 + 1 = y\}$. Then, the initial curve $\Gamma(s) = \{x_o(s), y_0(s), \tilde{u}_0(s)\}$ needed to start applying the Method of Characteristic is given by

◯ $\{s^3 + 1, s, 2\}$
X $\{s, s^3 + 1, 2\}$
X $\{s^{1/3}, s + 1, 2\}$
◯ $\{s + 1, s^3, 2\}$

Extra exercises

2.4. Find a solution Consider the PDE

$$xu_x + yu_y = -2u.$$

Find a solution to the previous PDE such that $u \equiv 1$ on the unit circle.

SOL: The condition for $u \equiv 1$ on the unit circle can be translated into $u(\cos(s), \sin(s)) = 1$ for $0 \leq s < 2\pi$. Thus, the characteristic equations and parametric initial conditions are given by

$$\begin{aligned} &x_t(t,s) = x, && y_t(t,s) = y, && u_t(t,s) = -2u, \\ &x(0,s) = \cos(s), && y(0,s) = \sin(s), && u(0,s) = 1. \end{aligned}$$

Solving each of the ODEs separately, we obtain that

$$x(t,s) = \cos(s)e^t, \quad y(t,s) = \sin(s)e^t, \quad u(t,s) = e^{-2t}.$$

Now notice that $x^2 + y^2 = e^{2t} = \frac{1}{u}$. Therefore,

$$u(x,y) = \frac{1}{x^2 + y^2}$$

is a solution to the PDE, defined for $(x,y) \neq (0,0)$.