1.1. Classification of PDEs Determine the order of the following PDEs. Determine also whether they are linear or not. If they are linear, determine if they are homogeneous or not, and if they are not linear, determine if they are quasilinear.

(a) $\cos(x+y)u_x - \sin(y)u_{yy} = 0$.

(b) $\Delta(u - u_y) = 4$.

(c) $(u_{xx}+1)^3 = x^3 + 2$.

(d) $u_{xx}u_{yy} - (u_{xy})^2 = 1$.

(e) $u_{xy} - u_x u_y = 2$.

SOL:

(a) Second order, linear, homogeneous.

(b) Third order, linear, inhomogeneous.

(c) Second order, linear, inhomogeneous. To see this, notice that this PDE is equivalent to $u_{xx} = (x^3+2)^{\frac{1}{3}} - 1$.

(d) Second order, not linear. This is not quasilinear.

(e) Second order, not linear. It is quasilinear.

1.2. Solutions to ODEs Solve the following ODEs.

(a) $x'(t) + \lambda x(t) = 0$, with $x(0) = x_0$.

(b) $x'(t) + \lambda x(t) = 1$, with $x(0) = x_0$.

(c) $x'(t) + x(t) = t$, with $x(0) = 1$.

(d) $x'(t) + x(t) = e^t$, with $x(0) = 1$.

(e) $x''(t) + \lambda^2 x(t) = 0$, find a general solution.

SOL:

(a) Dividing both sides by $x(t)$ and integrating we get

$$\int_0^t \frac{x'(s)}{x(s)}\, ds = -\lambda t,$$

and since $\frac{x'(s)}{x(s)} = \frac{d}{ds}\ln(x(s))$, we get $\ln(x(t)) - \ln(x_0) = -\lambda t$, obtaining $x(t) = e^{\ln(x_0)-\lambda t} = x_0 e^{-\lambda t}$.

(b) Multiplying the ODE by $e^{\lambda t}$ we get[1]

$$e^{\lambda t}x'(t) + \lambda e^{\lambda t}x(t) = e^{\lambda t}.$$

Now, note that the left hand side is equal to $\frac{d}{dt}(e^{\lambda t}x(t))$, so integrating

$$\int_0^t \frac{d}{ds}(e^{\lambda s}x(s))\,ds = \int_0^t e^{\lambda s}\,ds,$$

one gets

$$e^{\lambda t}x(t) - x_0 = \frac{1}{\lambda}(e^{\lambda t} - 1).$$

Rearranging we finally get $x(t) = \frac{1}{\lambda}(1 - e^{-\lambda t}) + x_0 e^{-\lambda t}$.

(c) Multiplying by e^t, we get

$$e^t x'(t) + e^t x(t) = te^t,$$

and since the left hand side of this expression is equal to $\frac{d}{dt}(e^t x(t))$, we can integrate, obtaining

$$e^t x(t) - 1 = \int_0^t se^s\,ds = [se^s]_0^t - \int_0^t e^s\,ds = te^t - e^t + 1,$$

where we integrated by parts the left hand side. Rearranging,

$$x(t) = t - 1 + 2e^{-t}.$$

(d) Again, multiplying by e^t and integrating we get

$$e^t x(t) - 1 = \int_0^t e^{2s}\,dt = \frac{1}{2}(e^{2t} - 1),$$

obtaining

$$x(t) = \frac{1}{2}(e^t + e^{-t}) = \cosh(t).$$

(e) The general solution of this (very important!) ODE is given by

$$x(t) = A\sin(\lambda t) + B\cos(\lambda t),$$

where A and B are arbitrary constants.

[1] In general, for first order linear ODE of the from $x'(t) + a(t)x(t) = b(t)$, the trick is to multiply everything by $e^{A(t)}$, where $A'(t) = a(t)$, so that $\frac{d}{dt}(e^{A(t)}x(t)) = e^{A(t)}b(t)$. The solution follows by integration.

1.3. Nonexistence of solutions Show that there is not a smooth function $u : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{cases} u_x = xy\,, \\ u_y = x^2\,. \end{cases}$$

SOL: Differentiating the first PDE in the y direction we get $u_{yx} = x$ and differentiating the second PDE in the x direction we get $u_{xy} = 2x$. By the Schwarz theorem, the partial derivatives of a smooth function commute, in the sense $u_{xy} = u_{yx}$. But this implies that $2x = u_{xy} = u_{yx} = x$, which is absurd for all $x \neq 0$, contradicting the Schwarz theorem, and implying that no smooth u solves the above PDE.

1.4. Existence of infinite solutions Consider the Cauchy problem (PDE + imposed initial data)

$$\begin{cases} u_x + u_y = 0, & (x,y) \in \mathbb{R}^2, \\ u(x,x) = 1, & x \in \mathbb{R}. \end{cases}$$

(a) Show that there exists an infinite amount of smooth functions $u : \mathbb{R}^2 \to \mathbb{R}$ solving it.

(b) Find a solution u such that for all $(x,y) \in \mathbb{R}^2$, $u(x,y) = u(y,x)$, $u(x,y) > 0$ and $\lim_{x\to+\infty} u(x,0) = 0$.

(c) Show that there is no alternating solution, i.e. satisfying $u(x,y) = -u(y,x)$ for all $(x,y) \in \mathbb{R}^2$.

SOL:

(a) There are many ways to solve this point. A simple option is to try with polynomials of order one $u(x,y) = ax + by + c$, obtaining the system

$$\begin{cases} 0 = u_x + u_y = a + b, \\ 1 = u(x,x) = (a+b)x + c, \end{cases}$$

from the first equation we get $a = -b$, and from the second $c = 1$, so that $u(x,y) = a(x-y)+1$. Notice that this polynomial solves the PDE for all $a \in \mathbb{R}$, giving a whole family of distinct solutions.

(b) Notice that $u(x,y) = f(x-y)$ is a solution of the PDE for all C^1-function $f : \mathbb{R} \to \mathbb{R}$ so that $f(0) = 1$. In fact,

$$\begin{cases} u_x + u_y = f'(x-y) - f'(x-y) = 0, \\ u(x,x) = f(x-x) = f(0) = 1. \end{cases}$$

Therefore, we need to find f such that $f(x-y) = f(y-x)$, that is $f(s) = f(-s)$ for all $s \in \mathbb{R}$, $f(0) = 1$ and $0 = \lim_{x\to+\infty} u(x,0) = \lim_{x\to+\infty} f(x)$. There are many functions satisfying this three conditions. For example $f(s) = e^{-s^2}$.

(c) If there where, swapping x with x we get in particular $u(x,x) = -u(x,x)$ for all $x \in \mathbb{R}$. This is not possible because $u(x,x) = 1$, and $1 \neq -1$.

1.5. Multiple Choice Determine correct answer(s) to each point.

(a) Let $A : \mathbb{R}^4 \to \mathbb{R}$ be a nontrivial ($A \not\equiv 0$) linear map. Then, the PDE $A(u, u_x, u_y, u_{xx}) = 0$ is *always*

X linear	◯ of second order
◯ inhomogeneous	◯ of first order
X homogeneous	◯ quasilinear

SOL: The right answers follow by the very definition of being linear and homogeneuos. If it is homogeneous, it cannot be inhomogeneous. It can be of order lesser than 2, for example $A(u, u_x, u_y, u_{xx}) = u_x + 2u_y$, but it can be of order two for example $A(u, u_x, u_y, u_{xx}) = u_{xx} + 2u_y$. It is linear, so it cannot be quasilinear.

(b) The reaction-diffusion equation (widely used in biology, geology, chemistry, physics and ecology as a model for pattern formation) takes this form in its simple formulation: $u_t = \Delta u + R(u)$, where $R : \mathbb{R} \to \mathbb{R}$ accounts for local reactions, and $u = u(t,x,y)$ represents the unknown (density of a chemical substance/population etc) at time $t \geq 0$ and position $(x,y) \in \mathbb{R}^2$. This PDE is

◯ of third order	◯ inhomogeneous if R is a linear map
X of second order	◯ fully nonlinear
X linear if R is an affine map	◯ of first order

SOL: The Laplacian Δ is an order two operator. If R is affine, that is $R(u) = au + b$, then the PDE is clearly linear. If R is linear, that is $R(u) = au$, then the PDE is homogeneous.

(c) Consider the PDE $\Delta u + \nabla u \cdot \nabla u = 0$, where with the "dot" we denote the usual scalar product in $\mathbb{R}^n$. Then, the function $v := e^u$ solves a PDE that is

X linear and homogeneous ◯ quasilinear

◯ linear and inhomogeneous ◯ fully nonlinear

SOL: Substituting $u = \ln(v)$ in the PDE for u we get

$$\begin{aligned} 0 &= \Delta(\ln(v)) + \nabla(\ln(v)) \cdot \nabla(\ln(v)) = (\ln(v))_{xx} + (\ln(v))_{yy} + ((\ln(v))_x)^2 + ((\ln(v))_y)^2 \\ &= \left(\frac{v_x}{v}\right)_x + \left(\frac{v_y}{v}\right)_y + \left(\frac{v_x}{v}\right)^2 + \left(\frac{v_y}{v}\right)^2 \\ &= \frac{\Delta v}{v} - \frac{v_x^2 + v_y^2}{v^2} + \left(\frac{v_x}{v}\right)^2 + \left(\frac{v_y}{v}\right)^2 = \frac{\Delta v}{v}. \end{aligned}$$

Hence, v solves $\frac{\Delta v}{v} = 0$, that is equivalent (by multiplying by v) to $\Delta v = 0$.

(d) For a smooth vector field $F = (F^1, F^2, \ldots, F^n) : \mathbb{R}^n \to \mathbb{R}^n$, we denote with $\operatorname{div}(F)$ the divergence of F, given by $\operatorname{div}(F) := \sum_{i=1}^n F^i_{x_i}$. The following PDE

$$\operatorname{div}(\nabla(u^2)) = u,$$

is

◯ of third order and fully nonlinear X quasilinear

◯ linear X of second order

SOL : If the dimension is confusing, try to solve this for $n = 2$. The answer is the same. We have

$$\begin{aligned} u &= \operatorname{div}(\nabla(u^2)) = \operatorname{div}(2u\nabla u) = \sum_{i=1}^n (2uu_{x_i})_{x_i} = 2\sum_{i=1}^n \Big((u_{x_i})^2 + uu_{x_ix_i}\Big) \\ &= 2\nabla u \cdot \nabla u + 2u\Delta u. \end{aligned}$$

The PDE $2u\Delta u + 2\nabla u \cdot \nabla u - u = 0$ is quasilinear and of second order since the Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is a linear second order operator.

Extra exercises

1.6. Classification of PDEs, II Determine the order of the following PDEs. Determine also whether they are linear or not. If they are linear, determine if they are homogeneous or not, and if they are not linear, determine if they are quasilinear.

(a) $\Delta(\Delta u) = 5u$.

(b) $10^{20}u + \sin(u_x) = u_{xx}$.

(c) $e^{\Delta u} = u$.

(d) $\partial_x(uu_y) = \partial_y(uu_x)$.

SOL:

(a) $\Delta(\Delta u) = 5u$.

Fourth order. Linear. Homogeneous.

(b) $10^{20}u + \sin(u_x) = u_{xx}$

Second order. Not linear. Quasilinear.

(c) $e^{\Delta u} = u$.

The PDE can be rewritten as $\Delta u = \log(u)$. Second order. Not linear. Quasilinear.

(d) $\partial_x(uu_y) = \partial_y(uu_x)$.

Notice that $\partial_x(uu_y) = u_x u_y + uu_{xy}$ and $\partial_y(uu_x) = u_y u_x + uu_{xy}$. So the PDE is always true (at least for smooth functions) and should not be categorized as linear/not linear or homogeneous/not homogenous.

1.7. Solutions to PDEs Check whether each of the following PDEs has a solution u that is a polynomial and, if it exists, determine a polynomial that solves the PDE.

(a) $\Delta u = x + y$.

(b) $u_{xx} = -u$, with $u(0) = 1$.

(c) $u_{xx} + u_{xy} = \sin(x)$.

(d) $u_{xyx}^2 + u_{yxy} = e^u$.

(e) $u_{xx} + u_y + u_{xy} = x^2 y$.

SOL:

(a) $\Delta u = x + y$.

The polynomial $u(x, y) = \frac{1}{6}x^3 + \frac{1}{6}y^3$ solves the PDE.

(b) $u_{xx} = -u$, with $u(0) = 1$.

The general solution of the ODE $u_{xx} = -u$ is $u(x) = \alpha \sin(x) + \beta \cos(x)$. Hence, since $u(0) = 1$, it holds $u(x) = \alpha \sin(x) + \cos(x)$, which is not a polynomial for any choice of $\alpha \in \mathbb{R}$.

(c) $u_{xx} + u_{xy} = \sin(x)$.

If u is a polynomial, then $u_{xx} + u_{xy}$ is a polynomial. Therefore, since $\sin(x)$ is not a polynomial, there is not a solution u which is a polynomial.

(d) $u_{xyx}^2 + u_{yxy} = e^u$.

If u is a polynomial then $u_{xyx}^2 + u_{yxy}$ is a polynomial. Hence, if u solves and is a polynomial, then e^u is a polynomial. But if u and e^u are both polynomial then u must be constant (think of the growth as x goes to infinity). Since $u \equiv \text{const}$ is not a solution of the PDE, there PDE does not have a polynomial solution.

(e) $u_{xx} + u_y + u_{xy} = x^2 y$.

The polynomial $u(x, y) = \frac{1}{2}x^2y^2 - \frac{y^3}{3} - xy^2 + y^2$ solves the PDE.