13.1. Harmonic function in the disk

Let $D := \{x^2 + y^2 < 1\}$. Find the solution to the following problem

$$\begin{cases} \Delta u &= 0, & \text{for } (x,y) \in D, \\ u(x,y) &= x^3 + x, & \text{for } (x,y) \in \partial D. \end{cases}$$

Hint: It holds $\cos(\theta)^3 = \frac{1}{4}(3\cos(\theta) + \cos(3\theta))$.

13.2. Harmonic function in the annulus

Find the solution to the following problem, posed for $2 < r < 4$ and $-\pi < \theta \leq \pi$:

$$\begin{cases} \Delta u &= 0, & \text{for } 2 < r < 4, \\ u(2,\theta) &= 0, & \text{for } -\pi < \theta \leq \pi, \\ u(4,\theta) &= \sin(\theta), & \text{for } -\pi < \theta \leq \pi. \end{cases}$$

13.3. Big on the boundary, small inside

Let $B_r := \{x^2 + y^2 < r\}$ be the ball centered at the origin with radius $r > 0$. Find a harmonic function $u : \bar{B}_1 \to \mathbb{R}$ such that

$$|u| < 0.00001 \text{ in } B_{\frac{1}{2}} \quad \text{and} \quad \int_{\partial B_1} |u| > 1000\,.$$