

8.1. Separation of variables

Solve the following equations using the method of separation of variables and superposition principle. To do so, write first a general solution solving the problem with boundary conditions, and then impose the initial values.

(a)

$$\begin{cases} u_t - u_{xx} = 0, & (x, t) \in (0, \pi) \times (0, \infty), \\ u(0, t) = 0, & t \in (0, \infty), \\ u(\pi, t) = 0, & t \in (0, \infty), \\ u(x, 0) = \sin(2x) + \frac{1}{2} \sin(3x) + 5 \sin(5x), & x \in [0, \pi]. \end{cases}$$

(b)

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x, t) \in (0, \pi) \times (0, \infty), \\ u(0, t) = 0, & t \in (0, \infty), \\ u(\pi, t) = 0, & t \in (0, \infty), \\ u(x, 0) = \sin^3(x), & x \in [0, \pi], \\ u_t(x, 0) = \sin(2x), & x \in [0, \pi]. \end{cases}$$

Hint: recall that $4 \sin^3(x) = 3 \sin(x) - \sin(3x)$.

(c)

$$\begin{cases} u_t - u_{xx} = 0, & (x, t) \in (0, \pi) \times (0, \infty), \\ u_x(0, t) = 0, & t \in (0, \infty), \\ u_x(\pi, t) = 0, & t \in (0, \infty), \\ u(x, 0) = 1 + \cos(x) & x \in [0, \pi]. \end{cases}$$

8.2. Multiple choice Cross the correct answer(s).

(a) Let u be solution of the heat equation

$$\begin{cases} u_t - k u_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty), \\ u(0, t) = u(L, t) = 0, & t > 0, \\ u(x, 0) = f(x), & x \in (0, L). \end{cases}$$

for $f \in C^\infty(0, T)$. Then, for all $a > 0$

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| ☐ $\lim_{t \rightarrow +\infty} \int_0^L u(x, t)^2 dx = +\infty$ | ☐ $\lim_{t \rightarrow +\infty} t^a \int_0^L u(x, t)^2 dx = 0$ |
| ☐ $\lim_{t \rightarrow +\infty} \int_0^L u(x, t)^2 dx = 0$ | ☐ $\lim_{t \rightarrow +\infty} t^a \int_0^L u(x, t)^2 dx = +\infty$ |

(b) Consider the periodic homogeneous wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & (x, t) \in [0, 1] \times [0, +\infty) \\ u_x(0, t) = u_x(1, t) = 0, & t > 0, \\ u(x, 0) = 1 + 2021 \cos(2\pi x), & x \in [0, 1], \\ u_t(x, 0) = \cos(40\pi x), & x \in [0, 1]. \end{cases}$$

Then, for a fixed point $\bar{x} \in [0, 1]$, the function $t \mapsto u(\bar{x}, t)$ has period

☐ $1/40$

☐ $1/2$

☐ 2π

☐ π