2.1. Method of characteristics I Solve the following equations using the method of characteristics.

(a) $u_x + u_y = 1$, with $u(x, 0) = f(x)$.

(b) $xu_x + (x + y)u_y = 1$, with $u(1, y) = y^2$.

(c) $u_x - 2xyu_y = 0$, with $u(0, y) = y$.

(d) $yu_x - xu_y = 0$, with $u(x, 0) = g(x^2)$ for all $x > 0$.

2.2. Method of characteristics II Consider the PDE

$$u_x + (x + y)u_y = 1.$$

Solve the general system of ODEs associated to this PDE. Then, for each initial data listed below, find an explicit solution via the Method of Characteristics if possible. If it is not possible, explain why.

(a) $u(0, y) = 1 - y$.

(b) $u(x, -1 - x) = e^x$, $x \in \mathbb{R}$.

2.3. Multiple choice Cross the correct answer(s).

(a) The expression $f(u_{xxx}) = u_z + 5$ describes a quasilinear PDE of order 3 if

- ◯ f is linear
- ◯ f is constant
- ◯ f is invertible
- ◯ f is a polynomial

(b) The Hessian of a C^2-function $u : \mathbb{R}^n \to \mathbb{R}$ is the $n \times n$ symmetric matrix D^2u, whose coefficients are $(D^2u)_{ij} = u_{x_i x_j}$, $i, j \in \{1, \dots, n\}$. For $n \geq 2$ the *Monge-Ampère equations* are the PDEs in the form: $\det(D^2u) = f$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a given smooth function. These PDEs are

- ◯ fully nonlinear
- ◯ linear inhomogeneous if $f \not\equiv 0$
- ◯ quasilinear
- ◯ of second order
- ◯ linear homogeneous if $f \equiv 0$
- ◯ of third order

(c) Let $\Omega \subset \mathbb{R}^2$. Given a function $H : \Omega \to \mathbb{R}$, to find a function $u : \Omega \to \mathbb{R}$ whose surface graph $\Sigma = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$ has *mean curvature* equal to $H(x, y)$ at each point $(x, y, u(x, y)) \in \Sigma$, one has to solve the *prescribed mean curvature equation*:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = H.$$

Here $\nabla u = (u_x, u_y)$ and $|\nabla u|^2 = (u_x)^2 + (u_y)^2$. This PDE is

- ◯ fully nonlinear
- ◯ quasilinear
- ◯ linear homogeneous if $H \equiv 0$
- ◯ linear inhomogeneous if $H \not\equiv 0$
- ◯ of second order
- ◯ of third order

(d) Consider the PDE $yu_x - x^2 u_y = 0$ coupled with the boundary condition $u(x, y) = 2$ on $\{(x, y) : x^3 + 1 = y\}$. Then, the initial curve $\Gamma(s) = \{x_o(s), y_0(s), \tilde{u}_0(s)\}$ needed to start applying the Method of Characteristic is given by

- ◯ $\{s^3 + 1, s, 2\}$
- ◯ $\{s, s^3 + 1, 2\}$
- ◯ $\{s^{1/3}, s + 1, 2\}$
- ◯ $\{s + 1, s^3, 2\}$

Extra exercises

2.4. Find a solution Consider the PDE

$$xu_x + yu_y = -2u.$$

Find a solution to the previous PDE such that $u \equiv 1$ on the unit circle.