

1.1. Classification of PDEs Determine the order of the following PDEs. Determine also whether they are linear or not. If they are linear, determine if they are homogeneous or not, and if they are not linear, determine if they are quasilinear.

(a) $\cos(x + y)u_x - \sin(y)u_{yy} = 0$.

(b) $\Delta(u - u_y) = 4$.

(c) $(u_{xx} + 1)^3 = x^3 + 2$.

(d) $u_{xx}u_{yy} - (u_{xy})^2 = 1$.

(e) $u_{xy} - u_xu_y = 2$.

1.2. Solutions to ODEs Solve the following ODEs.

(a) $x'(t) + \lambda x(t) = 0$, with $x(0) = x_0$.

(b) $x'(t) + \lambda x(t) = 1$, with $x(0) = x_0$.

(c) $x'(t) + x(t) = t$, with $x(0) = 1$.

(d) $x'(t) + x(t) = e^t$, with $x(0) = 1$.

(e) $x''(t) + \lambda^2 x(t) = 0$, find a general solution.

1.3. Nonexistence of solutions Show that there is not a smooth function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\begin{cases} u_x = xy, \\ u_y = x^2. \end{cases}$$

1.4. Existence of infinite solutions Consider the Cauchy problem (PDE + imposed initial data)

$$\begin{cases} u_x + u_y = 0, & (x, y) \in \mathbb{R}^2, \\ u(x, x) = 1, & x \in \mathbb{R}. \end{cases}$$

(a) Show that there exists an infinite amount of smooth functions $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ solving it.

(b) Find a solution u such that for all $(x, y) \in \mathbb{R}^2$, $u(x, y) = u(y, x)$, $u(x, y) > 0$ and $\lim_{x \rightarrow +\infty} u(x, 0) = 0$.

(c) Show that there is no alternating solution, i.e. satisfying $u(x, y) = -u(y, x)$ for all $(x, y) \in \mathbb{R}^2$.

1.5. Multiple Choice Determine correct answer(s) to each point.

(a) Let $A : \mathbb{R}^4 \rightarrow \mathbb{R}$ be a nontrivial ($A \not\equiv 0$) linear map. Then, the PDE $A(u, u_x, u_y, u_{xx}) = 0$ is *always*

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|-------------------------------------|---------------------------------------|
| <input type="radio"/> linear | <input type="radio"/> of second order |
| <input type="radio"/> inhomogeneous | <input type="radio"/> of first order |
| <input type="radio"/> homogeneous | <input type="radio"/> quasilinear |

(b) The reaction-diffusion equation (widely used in biology, geology, chemistry, physics and ecology as a model for pattern formation) takes this form in its simple formulation: $u_t = \Delta u + R(u)$, where $R : \mathbb{R} \rightarrow \mathbb{R}$ accounts for local reactions, and $u = u(t, x, y)$ represents the unknown (density of a chemical substance/population etc) at time $t \geq 0$ and position $(x, y) \in \mathbb{R}^2$. This PDE is

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| <input type="radio"/> of third order | <input type="radio"/> inhomogeneous if R is a linear map |
| <input type="radio"/> of second order | <input type="radio"/> fully nonlinear |
| <input type="radio"/> linear if R is an affine map | <input type="radio"/> of first order |

(c) Consider the PDE $\Delta u + \nabla u \cdot \nabla u = 0$, where with the "dot" we denote the usual scalar product in \mathbb{R}^n . Then, the function $v := e^u$ solves a PDE that is

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|--|---------------------------------------|
| <input type="radio"/> linear and homogeneous | <input type="radio"/> quasilinear |
| <input type="radio"/> linear and inhomogeneous | <input type="radio"/> fully nonlinear |

(d) For a smooth vector field $F = (F^1, F^2, \dots, F^n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we denote with $\operatorname{div}(F)$ the divergence of F , given by $\operatorname{div}(F) := \sum_{i=1}^n F_{x_i}^i$. The following PDE

$$\operatorname{div}(\nabla(u^2)) = u,$$

is

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|--|---------------------------------------|
| <input type="radio"/> of third order and fully nonlinear | <input type="radio"/> quasilinear |
| <input type="radio"/> linear | <input type="radio"/> of second order |

Extra exercises

1.6. Classification of PDEs, II Determine the order of the following PDEs. Determine also whether they are linear or not. If they are linear, determine if they are homogeneous or not, and if they are not linear, determine if they are quasilinear.

(a) $\Delta(\Delta u) = 5u$.

(b) $10^{20}u + \sin(u_x) = u_{xx}$.

(c) $e^{\Delta u} = u$.

(d) $\partial_x(uu_y) = \partial_y(uu_x)$.

1.7. Solutions to PDEs Check whether each of the following PDEs has a solution u that is a polynomial and, if it exists, determine a polynomial that solves the PDE.

(a) $\Delta u = x + y$.

(b) $u_{xx} = -u$, with $u(0) = 1$.

(c) $u_{xx} + u_{xy} = \sin(x)$.

(d) $u_{xyx}^2 + u_{yxy} = e^u$.

(e) $u_{xx} + u_y + u_{xy} = x^2y$.