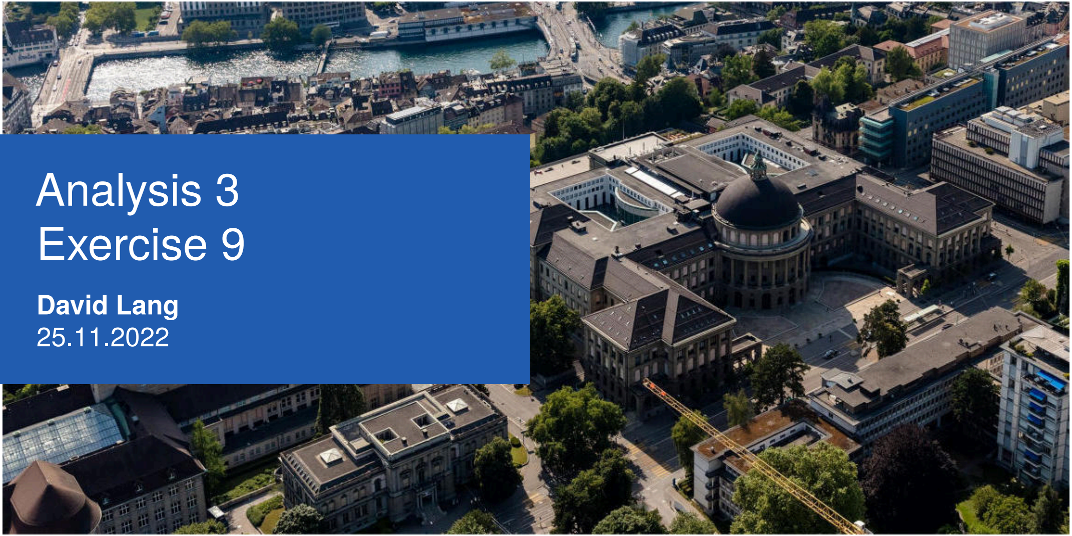

Analysis 3
Exercise 9

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Outline

1. Serie 8 Review
2. Course Overview
3. Separation of Variables for Homogeneous Equations
4. Separation of Variables for Inhomogeneous Equations
5. The energy method and uniqueness
6. Tips for Serie 9

Outline

Serie 8 Review

1. Separation of variables
   - (a) Don't forget to multiply $X(x)$ and $T(t)$ at the end.
     Don't use k as an iterative index, use m or n.
   - (b) Heat Equation: $T(t)$ is exponential decay; Wave equation: $T(t)$ is sum of sin and cos
     Apply chain rule while calculating $u_t(x, 0)$.
   - (c) If there is only a finite number of terms, please write down the solution explicitly.
2. Multiple choice
   - (a) Thermal conductivity k is defined as $\vec{q} = -k\nabla T$
     Fourier's Law of heat conduction
   - (b) $\cos(40\pi x) = A_0 + \sum_{n=1}^{\infty} 2\pi n D_n \cos(n\pi x)$ $D_{40} = \frac{1}{80\pi}$
     D'Alembert formula is only applicable for $x \in \mathbb{R}$

Outline

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - **Wave equation**
    - D'Alembert formula
    - **Separation of variables**
  - Parabolic PDEs
    - **Heat equation**
    - Maximum principle
    - **Separation of variables**
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

Separation of Variables for Homogeneous Equations

	Wave Equation	Heat Equation
Dirichlet	$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x) \left[A_n \cos(\frac{n\pi c}{L}t) + B_n \sin(\frac{n\pi c}{L}t)\right]$	$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}$
Nuemann	$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos(\frac{n\pi}{L}x) \left[A_n \cos(\frac{n\pi c}{L}t) + B_n \sin(\frac{n\pi c}{L}t)\right]$	$u(x,t) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}$

Outline

Procedure for inhomogeneous equations using separation of variables

1. $$u(x,t) = X(x)T(t)$$
2. Choose $X(x)$ according to the boundary condition, do not solve $T(t)$ yet.
   - (a) Dirichlet
   - (b) Neumann
   - (c) Mixed = Dirichlet + Neumann
   - (d) Mixed = Neumann + Dirichlet
   - (e) Nonhomognenous boundary condition
3. Plug in the general formulation $u(x,t)$ into the original PDE
4. Solve the ODE for $T_n(t)$ together with the initial condition
5. Write down the explicit solution.

Eigenfunctions according to boundary conditions

- Dirichlet $u(0,t) = u(L,t) = 0$

$$X_n(x) = \sin(\frac{n\pi}{L}x) \quad \lambda_n = (\frac{n\pi}{L})^2, \quad n = 1, 2, 3, ...$$

- Neumann $u_x(0,t) = u_x(L,t) = 0$

$$X_n(x) = \cos(\frac{n\pi}{L}x) \quad \lambda_n = (\frac{n\pi}{L})^2, \quad n = 0, 1, 2, ...$$

- Mixed = Dirichlet + Neumann $u(0,t) = u_x(L,t) = 0$

$$X_n(x) = \sin((n+\frac{1}{2})\frac{\pi}{L}x) \quad \lambda_n = ((n+\frac{1}{2})\frac{\pi}{L})^2, \quad n = 0, 1, 2, ...$$

- Mixed = Neumann + Dirichlet $u_x(0,t) = u(L,t) = 0$

$$X_n(x) = \cos((n+\frac{1}{2})\frac{\pi}{L}x) \quad \lambda_n = ((n+\frac{1}{2})\frac{\pi}{L})^2, \quad n = 0, 1, 2, ...$$

Inhomogeneous Heat equation with Dirichlet boundary conditions

Example 1

$$\begin{cases} u_t - ku_{xx} = \sin(3\pi x) & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = \sin(\pi x) & x \in (0,1) \end{cases}$$

$u(x,t) = X(x)T(t)$

Dirichlet $L=1$

$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) = \sin(n\pi x) \quad n = 1,2,3,\ldots$

$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$

$u_t - ku_{xx} = \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)\right) - k\frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)\right)$

$= \sum_{n=1}^{\infty} T_n'(t)\sin(n\pi x) + k n^2\pi^2 \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$

$$\begin{cases} u_t - ku_{xx} = \sin(3\pi x) & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = \sin(\pi x) & x \in (0,1) \end{cases}$$

$$\sum_{n=1}^{\infty} \left(T_n'(t) + kn^2\pi^2 T_n(t)\right) \sin(n\pi x) = \sin(3\pi x)$$

$$\begin{cases} T_3'(t) + 9k\pi^2 T_3(t) = 1 & n=3 \\ T_n'(t) + kn^2\pi^2 T_n(t) = 0 & n \neq 3 \end{cases}$$

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin(n\pi x) = \sin(\pi x)$$

$$\begin{cases} T_1(0) = 1 & n=1 \\ T_n(0) = 0 & n \neq 1 \end{cases}$$

Inhomogeneous Heat equation with Dirichlet boundary conditions
Example 1

$$\begin{cases} u_t - ku_{xx} = \sin(3\pi x) & (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = \sin(\pi x) & x \in (0,1) \end{cases}$$

$$\begin{cases} T_1'(t) + k\pi^2 T_1(t) = 0 & T_1(0) = 1 & n = 1 \\ T_3'(t) + 9k\pi^2 T_3(t) = 1 & T_3(0) = 0 & n = 3 \\ T_n'(t) + kn^2\pi^2 T_n(t) = 0 & T_n(0) = 0 & n \neq 1, 3 \end{cases}
\qquad
\begin{cases} T_1(t) = e^{-\pi^2 kt} & n = 1 \\ T_3(t) = \frac{1}{9\pi^2 k}\left(1 - e^{-9\pi^2 kt}\right) & n = 3 \\ T_n(t) = 0 & n \neq 1, 3 \end{cases}$$

$$u(x,t) = e^{-\pi^2 kt} \sin(\pi x) + \frac{1}{9\pi^2 k}\left(1 - e^{-9\pi^2 kt}\right) \sin(3\pi x)$$

Inhomogeneous Heat equation with Mixed boundary conditions
Example 2

$$\begin{cases} u_t - u_{xx} = \sin(9x/2) & (x,t) \in (0,\pi) \times (0,\infty) \\ u(0,t) = 0 & t > 0 \\ u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin(3x/2) & x \in (0,\pi) \end{cases}$$

$u(x,t) = X(x)T(t)$

Mixed $L = \pi$

$X_n(x) = \sin((n+\frac{1}{2})\frac{\pi}{L}x) = \sin((n+\frac{1}{2})x)$

$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\sin((n+\frac{1}{2})x)$

$$u_t - u_{xx} = \frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} T_n(t)\sin((n+\tfrac{1}{2})x)\right) - \frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} T_n(t)\sin((n+\tfrac{1}{2})x)\right)$$

$$= \sum_{n=0}^{\infty} T_n'(t)\sin((n+\tfrac{1}{2})x) + (n+\tfrac{1}{2})^2 \sum_{n=0}^{\infty} T_n(t)\sin((n+\tfrac{1}{2})x)$$

Inhomogeneous Heat equation with Mixed boundary conditions

Example 2

$$\begin{cases} u_t - u_{xx} = \sin(9x/2) & (x,t) \in (0,\pi) \times (0,\infty) \\ u(0,t) = 0 & t > 0 \\ u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin(3x/2) & x \in (0,\pi) \end{cases}$$

$$\sum_{n=0}^{\infty} \left(T_n'(t) + (n+\tfrac{1}{2})^2 T_n(t) \right) \sin\left((n+\tfrac{1}{2})x\right) = \sin\left(\tfrac{9}{2}x\right)$$

$$\begin{cases} T_4'(t) + \frac{81}{4} T_4(t) = 1 & n = 4 \\ T_n'(t) + (n+\frac{1}{2})^2 T_n(t) = 0 & n \neq 4 \end{cases}$$

$$u(x,0) = \sum_{n=0}^{\infty} T_n(0) \sin\left((n+\tfrac{1}{2})x\right) = \sin\left(\tfrac{3}{2}x\right)$$

$$\begin{cases} T_1(0) = 1 & n = 1 \\ T_n(0) = 0 & n \neq 1 \end{cases}$$

Inhomogeneous Heat equation with Mixed boundary conditions
Example 2

$$\begin{cases} u_t - u_{xx} = \sin(9x/2) & (x,t) \in (0,\pi) \times (0,\infty) \\ u(0,t) = 0 & t > 0 \\ u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin(3x/2) & x \in (0,\pi) \end{cases}$$

$$\begin{cases} T_1'(t) + \frac{9}{4} T_1(t) = 0 & T_1(0) = 1 & n = 1 \\ T_4'(t) + \frac{81}{4} T_4(t) = 0 & T_4(0) = 0 & n = 4 \\ T_n'(t) + (n+\frac{1}{2})^2 T_n(t) = 0 & T_n(0) = 0 & n \neq 1,4 \end{cases}$$

$$\begin{cases} T_1(t) = e^{-\frac{9}{4}t} & n = 1 \\ T_4(t) = \frac{4}{81} - \frac{4}{81} e^{-\frac{81}{4}t} & n = 4 \\ T_n(t) = 0 & n \neq 1,4 \end{cases}$$

$$u(x,t) = e^{-\frac{9}{4}t} \sin(\tfrac{3}{2}x) + \frac{4}{81}\left(1 - e^{-\frac{81}{4}t}\right) \sin(\tfrac{9}{2}x)$$

Nonhomogeneous boundary conditions

	Boundary condition	$w(x,t)$
Dirichlet	$u(0,t) = a(t), u(L,t) = b(t)$	$w(x,t) = a(t) + \frac{x}{L}[b(t) - a(t)]$
Nuemann	$u_x(0,t) = a(t), u_x(L,t) = b(t)$	$w(x,t) = xa(t) + \frac{x^2}{2L}[b(t) - a(t)]$
Mixed	$u(0,t) = a(t), u_x(L,t) = b(t)$	$w(x,t) = a(t) + xb(t)$
Mixed	$u_x(0,t) = a(t), u(L,t) = b(t)$	$w(x,t) = (x - L)a(t) + b(t)$

The Function $v(x,t) = u(x,t) - w(x,t)$ satisfies the homogeneous boundary conditions.

Inhomogeneous Wave equation with nonhomogeneous boundary conditions

Example 3

$$\begin{cases} u_{tt} - u_{xx} = \sin(x) & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = \sin(2x) & x \in (0,\pi) \\ u_t(x,0) = \sin(3x) + \frac{x}{\pi} & x \in (0,\pi) \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) = t & t > 0 \end{cases}$$

Dirichlet with $a(t)=0$ $b(t)=t$ $L=\pi$ $w(x,t) = a(t) + \frac{x}{L}[b(t)-a(t)] = \frac{xt}{\pi}$

$v(x,t) = u(x,t) - w(x,t) = u(x,t) - \frac{xt}{\pi}$ satisfies the homogeneous boundary condition

$$\begin{cases} v_{tt} - v_{xx} = \sin(x) \\ v(x,0) = u(x,0) - w(x,0) = \sin(2x) \\ v_t(x,0) = u_t(x,0) - w_t(x,0) = \sin(3x) \\ v(0,t) = u(0,t) - w(0,t) = 0 \\ v(\pi,t) = u(\pi,t) - w(\pi,t) = 0 \end{cases}$$

$v(x,t) = X(x)T(t)$

homogeneous Dirichlet & $L=\pi$

$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) = \sin(nx)$ $n=1,2,3,\dots$

$v(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(nx)$

Inhomogeneous Wave equation with nonhomogeneous boundary conditions

Example 3

$$\begin{cases} u_{tt} - u_{xx} = \sin(x) & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = \sin(2x) & x \in (0,\pi) \\ u_t(x,0) = \sin(3x) + \frac{x}{\pi} & x \in (0,\pi) \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) = t & t > 0 \end{cases}$$

$$v_{tt} - v_{xx} = \frac{\partial^2}{\partial t^2}\left(\sum_{n=1}^{\infty} T_n(t)\sin(nx)\right) - \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} T_n(t)\sin(nx)\right)$$

$$= \sum_{n=1}^{\infty} \left(T_n''(t) + n^2 T_n(t)\right)\sin(nx) = \sin(nx)$$

$$\begin{cases} T_1''(t) + T_1(t) = 1 & n=1 \\ T_n''(t) + n^2 T_n(t) = 0 & n \neq 1 \end{cases}$$

$$v(x,0) = \sum_{n=1}^{\infty} T_n(0)\sin(nx) = \sin(2x)$$

$$\begin{cases} T_2(0) = 1 & n=2 \\ T_n(0) = 0 & n \neq 2 \end{cases}$$

$$v_t(x,0) = \sum_{n=1}^{\infty} T_n'(0)\sin(nx) = \sin(3x)$$

$$\begin{cases} T_3'(0) = 1 & n=3 \\ T_n'(0) = 0 & n \neq 3 \end{cases}$$

Inhomogeneous Wave equation with nonhomogeneous boundary conditions

Example 3

$$
\begin{cases}
u_{tt} - u_{xx} = \sin(x) & (x,t) \in (0,\pi) \times (0,\infty) \\
u(x,0) = \sin(2x) & x \in (0,\pi) \\
u_t(x,0) = \sin(3x) + \frac{x}{\pi} & x \in (0,\pi) \\
u(0,t) = 0 & t > 0 \\
u(\pi,t) = t & t > 0
\end{cases}
$$

$$
\begin{cases}
T_1''(t) + T_1(t) = 1 & T_1(0) = 0 & T_1'(0) = 0 & n=1 \\
T_2''(t) + 4T_2(t) = 0 & T_2(0) = 1 & T_1'(0) = 0 & n=2 \\
T_3''(t) + 9T_3(t) = 0 & T_3(0) = 0 & T_3'(0) = 1 & n=3 \\
T_n''(t) + n^2 T_n(t) = 0 & T_n(0) = 0 & T_n'(0) = 0 & n \geq 4
\end{cases}
\qquad
\begin{cases}
T_1(t) = 1 - \cos(t) & n=1 \\
T_2(t) = \cos(2t) & n=2 \\
T_3(t) = \frac{1}{3}\sin(3t) & n=3 \\
T_n(t) = 0 & n \geq 4
\end{cases}
$$

$$
u(x,t) = w(x,t) + v(x,t) = \frac{xt}{\pi} + (1-\cos(t))\sin(x) + \cos(2t)\sin(2x) + \frac{1}{3}\sin(3t)\sin(3x)
$$

Outline

The energy method and uniqueness

Prove the uniqueness of the solution of initial boundary value problems.

The method is based on the physical principle of energy conservation, although in some applications the object we refer to mathematically as an 'energy' is not necessarily the actual energy of a physical system.

Outline:

We want to show that the difference of two possible solutions is necessarily the zero solution.

Define an energy integral that is nonnegative and is a nonincreasing function of the time t.

In addition, for $t = 0$ the energy is zero and therefore, the energy is zero for all $t \geq 0$.

Due to the positivity of the energy, and the zero initial and boundary conditions it will follow that that the solution is zero.

The energy method and uniqueness for wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in (0,L) \times (0,\infty) \\ u_x(0,t) = u_x(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in (0,L) \\ u_t(x,0) = g(x) & x \in (0,L) \end{cases} \quad \begin{cases} w_{tt} - c^2 w_{xx} = 0 & (x,t) \in (0,L) \times (0,\infty) \\ w_x(0,t) = w_x(L,t) = 0 & t > 0 \\ w(x,0) = 0 & x \in (0,L) \\ w_t(x,0) = 0 & x \in (0,L) \end{cases}$$

$$w := u_1 - u_2$$

Define the total energy of the solution w at time t as

$$E(t) := \frac{1}{2} \int_0^L (w_t^2 + c^2 w_x^2)\, dx$$

$$E'(t) = \frac{d}{dt} \left[\frac{1}{2} \int_0^L (w_t^2 + c^2 w_x^2)\, dx \right] = \int_0^L (w_t w_{tt} + c^2 w_x w_{xt})\, dx$$

$$c^2 w_x w_{xt} = c^2 \left[\frac{\partial}{\partial x}(w_x w_t) - w_{xx} w_t \right] = c^2 \frac{\partial}{\partial x}(w_x w_t) - w_{tt} w_t$$

The energy method and uniqueness for wave equation

$$E'(t) = c^2 \int_0^L \frac{\partial}{\partial x}(w_x w_t)\, dx = c^2 (w_x w_t)|_0^L$$

The boundary condition implies that $E'(t) = 0$, hence $E(t) =$ constant and the energy is conserved.

One the other hand, since for $t = 0$ we have $w(x, 0) = 0$, it follows that $w_x(x, 0) = 0$.

Moreover, we have also $w_t(x, 0) = 0$.

Therefore, the energy at time $t = 0$ is zero.

Thus, $E(t) \equiv 0$.

Since $e(x, t) := w_t^2 + c^2 w_x^2 \geq 0$, and since its integral over $[0, L]$ is zero, it follows that $w_t^2 + c^2 w_x^2 \equiv 0$, which implies that $w_t(x, t) = w_x(x, t) \equiv 0$.

Consequently, $w(x, t) \equiv$ constant.

By the initial conditions $w(x, 0) = 0$, hence $w(x, t) \equiv 0$.

This completes the proof of the uniqueness of the problem.

The energy method and uniqueness for heat equation

$$\begin{cases} u_t - ku_{xx} = F(x,t) & (x,t) \in (0,L) \times (0,\infty) \\ u(0,t) = a(t) & t > 0 \\ u(L,t) = b(t) & t > 0 \\ u(x,0) = f(x) & x \in (0,L) \end{cases} \qquad \begin{cases} w_t - kw_{xx} = 0 & (x,t) \in (0,L) \times (0,\infty) \\ w(0,t) = 0 & t > 0 \\ w(L,t) = 0 & t > 0 \\ w(x,0) = 0 & x \in (0,L) \end{cases}$$

$$w := u_1 - u_2$$

$$E(t) := \frac{1}{2} \int_0^L w^2 \, dx$$

$$E'(t) = \frac{d}{dt} \left(\frac{1}{2} \int_0^L w^2 \, dx \right) = \int_0^L w w_t \, dx \; = \int_0^L k w w_{xx} \, dx$$

$$E'(t) = kww_x|_0^L - \int_0^L k w_x^2 \, dx = - \int_0^L k w_x^2 \, dx \leq 0$$

Therefore, the energy is not increasing.
Since $E(0) = 0$ and $E(t) \geq 0$, it follows that $E \equiv 0$.
Consequently, for all $t \geq 0$ we have $w(x,t) \equiv 0$ and the uniqueness is proved.

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Tips for Serie 9

1. Separation of variables for non-homogeneous problems
   - (a) Follow the procedure.
   - (b) Check the table to find the suitable function to subtract.
   - (c) Derive the eigenfunction and eigenvalue for practice.
2. Multiple choice
   - (a) Separation of variable leads to $T_n'(t) + (n^2 - p(t))T_n(t) = 0$.
     Then find the general solution
3. Extra exercises
   - Case study on n to find the $T_n(t)$.

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein