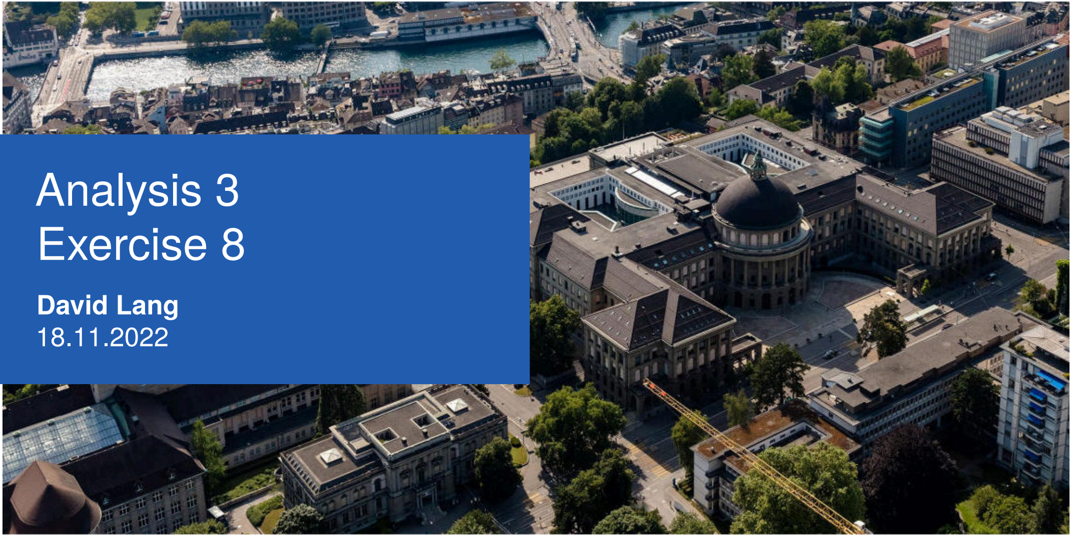

Analysis 3
Exercise 8

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18.11.2022

Outline

1. Serie 7 Review
2. Course Overview
3. Homogeneous Heat Equation
4. Homogeneous Wave Equation
5. Tips for Serie 8

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Serie 7 Review

1. (Non)homogeneous wave equation
   - Stick to the definition given in the problem.
   - For $|x-ct|<\alpha$ and $|x+ct|<\alpha$ etc.
2. Propagation of symmetries from initial data, II
   - $u_{tt} - 2u_{xx} = 0$ implies $c = \sqrt{2}$
3. Wave equation on a ring
   -
4. Multiple choice

$$g(x) = \sum_{n=0}^{N} b_n \cos(nx) \text{ and } \int_0^{2\pi} g(x)\, dx = 0, \text{ then } b_0 = \frac{1}{2\pi}\int_0^{2\pi} g(x)\, dx = 0$$

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5. Strange wave equation
   -

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - **Separation of variables**
  - Parabolic PDEs
    - **Heat equation**
    - Maximum principle
    - **Separation of variables**
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

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Homogeneous Heat equation with Dirichlet boundary conditions
Derived in lecture

$$\begin{cases} u_t - ku_{xx} = 0 & (x,t) \in (0,L) \times (0,\infty) \\ u(0,t) = u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in (0,L) \end{cases}$$

$$u(x,t) = X(x)T(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x)$$

$$A_m = \frac{2}{L} \int_0^L f(x) \sin(\frac{m\pi}{L}x)\, dx$$

Homogeneous Heat equation with Neumann boundary conditions

$$\begin{cases} u_t - ku_{xx} = 0 & (x,t) \in (0,L) \times (0,\infty) \\ u_x(0,t) = u_x(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in (0,L) \end{cases}$$

$$u(x,t) = X(x)T(t)$$

$$T'(t)X(x) - kX''(x)T(t) = 0$$

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$X''(x) = -\lambda X(x) \quad \text{and} \quad T'(t) = -\lambda k T(t)$$

$$X(x) = \begin{cases} \alpha \cosh(\sqrt{-\lambda}x) + \beta \sinh(\sqrt{-\lambda}x), & \lambda < 0 \\ \alpha + \beta x, & \lambda = 0 \\ \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x), & \lambda > 0 \end{cases}$$

Homogeneous Heat equation with Neumann boundary conditions

$$X'(x) = \begin{cases} \sqrt{-\lambda}\left[\alpha \sinh(\sqrt{-\lambda}x) + \beta \cosh(\sqrt{-\lambda}x)\right], & \lambda < 0 \\ \beta, & \lambda = 0 \\ \sqrt{\lambda}\left[-\alpha \sin(\sqrt{\lambda}x) + \beta \cos(\sqrt{\lambda}x)\right], & \lambda > 0 \end{cases}$$

If $\lambda < 0$, then $X(x) = 0$
If $\lambda = 0$, then $X(x) = \alpha$
If $\lambda > 0$, then $X_n(x) = \alpha_n \cos(\sqrt{\lambda_n}x)$ and $\lambda_n = (\frac{n\pi}{L})^2$

$$T'(t) = -\lambda k T(t)$$

$$T_n(t) = B_n e^{-k(\frac{n\pi}{L})^2 t}$$

$$u(x,t) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos(\frac{n\pi}{L}x) e^{-k(\frac{n\pi}{L})^2 t}$$

$$f(x) = u(x,0) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos(\frac{n\pi}{L}x)$$

$$B_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi}{L}x)\,dx$$

Example 1

Heat
Dirichlet
$L = \pi/2$

$$\begin{cases} u_t - k u_{xx} = 0 & (x,t) \in (0,\pi/2) \times (0,\infty) \\ u(0,t) = u(\pi/2,t) = 0 & t > 0 \\ u(x,0) = 3\sin(4x) & x \in (0,L) \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} = \sum_{n=1}^{\infty} A_n \sin(2nx) e^{-k4n^2 t}$$

$$u(x,0) = 3\sin(4x) = \sum_{n=1}^{\infty} A_n \sin(2nx)$$

$$\begin{cases} A_2 = 3 & n = 2 \\ A_n = 0 & n \neq 2 \end{cases}$$

$$u(x,t) = 3\sin(4x) e^{-16kt}$$

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Homogeneous Wave equation with Neumann boundary conditions
Derived in Lecture

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in (0,L) \times (0,\infty) \\ u_x(0,t) = u_x(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in (0,L) \\ u_t(x,0) = g(x) & x \in (0,L) \end{cases}$$

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos(\frac{n\pi}{L}x) \left[A_n \cos(\frac{n\pi c}{L}t) + B_n \sin(\frac{n\pi c}{L}t)\right]$$

$$f(x) = u(x,0) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{L}x)$$

$$A_m = \frac{2}{L}\int_0^L f(x)\cos(\frac{m\pi}{L}x)\,dx$$

$$g(x) = u_t(x,0) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\cos(\frac{n\pi}{L}x)$$

$$B_0 = \frac{2}{L}\int_0^L g(x)\,dx, \quad B_m = \frac{2}{cm\pi}\int_0^L g(x)\cos(\frac{m\pi}{L}x)\,dx$$

Example 2

Wave $c=2$

Neumann

$L=1$

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in (0,1) \times (0,\infty) \\ u_x(0,t) = u_x(1,t) = 0 & t > 0 \\ u(x,0) = \cos^2(\pi x) & x \in (0,1) \\ u_t(x,0) = \sin^2(\pi x)\cos(\pi x) & x \in (0,1) \end{cases}$$

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L} x\right)\left[A_n \cos\left(\frac{n\pi c}{L} t\right) + B_n \sin\left(\frac{n\pi c}{L} t\right)\right]$$

$$= \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos(n\pi x)\left[A_n \cos(2n\pi t) + B_n \sin(2n\pi t)\right]$$

$$u(x,0) = \cos^2(\pi x) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \cos(n\pi x)$$

$$\begin{cases} A_0 = 1 & n=0 \\ A_2 = \frac{1}{2} & n=2 \\ A_n = 0 & n \neq 0,2 \end{cases}$$

Example 2

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in (0,1) \times (0,\infty) \\ u_x(0,t) = u_x(1,t) = 0 & t > 0 \\ u(x,0) = \cos^2(\pi x) & x \in (0,1) \\ u_t(x,0) = \sin^2(\pi x)\cos(\pi x) & x \in (0,1) \end{cases}$$

$$u_t(x,0) = \sin^2(\pi x)\cos(\pi x) = \frac{1}{4}\cos(\pi x) - \frac{1}{4}\cos(3\pi x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cdot (2n\pi) \cdot \cos(n\pi x)$$

$$\begin{cases} B_1 = \frac{1}{8\pi} & n=1 \\ B_3 = -\frac{1}{24\pi} & n=3 \\ B_n = 0 & n \neq 1,3 \end{cases}$$

$$u(x,t) = \frac{1}{2} + \frac{1}{8\pi}\cos(\pi x)\sin(2\pi t) + \frac{1}{2}\cos(2\pi x)\cos(4\pi t) - \frac{1}{24\pi}\cos(3\pi x)\sin(6\pi t)$$

Homogeneous Wave equation with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in (0,L) \times (0,\infty) \\ u(0,t) = u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x) \left[A_n \cos(\frac{n\pi c}{L}t) + B_n \sin(\frac{n\pi c}{L}t) \right]$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x)$$

$$A_m = \frac{2}{L} \int_0^L f(x) \sin(\frac{m\pi}{L}x)\, dx$$

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin(\frac{n\pi}{L}x)$$

$$B_m = \frac{2}{cm\pi} \int_0^L g(x) \sin(\frac{m\pi}{L}x)\, dx$$

Example 3

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = 3\cos(x) & x \in \mathbb{R} \\ u_t(x,0) = 1 - \cos(4x) & x \in \mathbb{R} \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \end{cases}$$

$u(x,t) = X(x)T(t)$

$u_{tt} - 4u_{xx} = 0$

$X(x)T''(t) - 4X''(x)T(t) = 0$

$$\frac{T''(t)}{4T(t)} - \frac{X''(x)}{X(x)} = 0$$

$$\frac{T''(t)}{4T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$u_x(0,t) = u_x(\pi,t) = 0$

$X'(0)T(t) = X'(\pi)T(t) = 0$

$X'(0) = X'(\pi) = 0$ don't want the trivial solution

$$X(x) = \begin{cases} \alpha \sinh(\sqrt{-\lambda}x) + \beta\cosh(\sqrt{-\lambda}x) & \lambda < 0 \\ \alpha x + \beta & \lambda = 0 \\ \alpha \sin(\sqrt{\lambda}x) + \beta\cos(\sqrt{\lambda}x) & \lambda > 0 \end{cases}$$

Example 3

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = 3\cos(x) & x \in \mathbb{R} \\ u_t(x,0) = 1 - \cos(4x) & x \in \mathbb{R} \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \end{cases}$$

$$X'(x) = \begin{cases} \sqrt{-\lambda}[\alpha \sinh(\sqrt{-\lambda}x) + \beta \cosh(\sqrt{-\lambda}x)] & \lambda < 0 \\ \beta & \lambda = 0 \\ \sqrt{\lambda}[-\alpha \sin(\sqrt{\lambda}x) + \beta \cos(\sqrt{\lambda}x)] & \lambda > 0 \end{cases}$$

$$\begin{cases} X(x) = 0 & \lambda < 0 \\ X(x) = \alpha & \lambda = 0 \\ \begin{cases} x'(0) = \sqrt{\lambda}\beta = 0 & \Rightarrow \beta = 0 \\ x'(\pi) = -\sqrt{\lambda}\alpha \sin(\sqrt{\lambda}\pi) = 0 & \Rightarrow \sqrt{\lambda}\pi = n\pi \end{cases} & \lambda > 0 \end{cases}$$

$\Rightarrow \sqrt{\lambda}\pi = n\pi \quad \lambda = n^2$

$X_n(x) = \alpha_n \cos(\sqrt{\lambda_n})$

$= \alpha_n \cos(nx)$

$T''(t) = -4\lambda T(t) \quad \lambda = n^2$

$n = 0 \quad T''(t) = 0$

$T(t) = \dfrac{A_0 + B_0 t}{2}$

$n \neq 0 \quad T''(t) = -4n^2 T(t)$

$T(t) = A_n \cos(2nt) + B_n \sin(2nt)$

Example 3

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = 3\cos(x) & x \in \mathbb{R} \\ u_t(x,0) = 1 - \cos(4x) & x \in \mathbb{R} \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \end{cases}$$

$$u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos(nx)\left[A_n \cos(2nt) + B_n \sin(2nt)\right]$$

$$u(x,0) = 3\cos(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) \qquad u_t(x,0) = 1 - \cos(4x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cdot (2n) \cdot \cos(nx)$$

$$\begin{cases} n=1 & A_1 = 3 \\ n \neq 1 & A_n = 0 \end{cases} \qquad \begin{cases} B_0 = 2 & n=0 \\ B_4 = -\frac{1}{8} & n=4 \\ B_n = 0 & n \neq 0,4 \end{cases}$$

$$u(x,t) = t + 3\cos(x)\cos(2t) - \frac{1}{8}\cos(4x)\sin(8t)$$

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Tips for Serie 8

1. Separation of variables
   - I suggest repeating the derivation at least once. Consult the general solution only if you are fully familiar with the procedure.
   - (a)
   - (b) $4sin^3(x) = 3sin(x) - sin(3x)$
   - (c)
2. Multiple choice
   - (a) Fourier Series and Orthogonality
   - (b) Solve $u(x, t)$ explicitly and explore the trigonometric terms.

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More on Waves:

1. AT&T, Dr. Shive: https://youtu.be/DovunOxlY1k

More on Fourier Series:

1. Prof. Strang: https://youtu.be/vA9dfINW4Rg

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein