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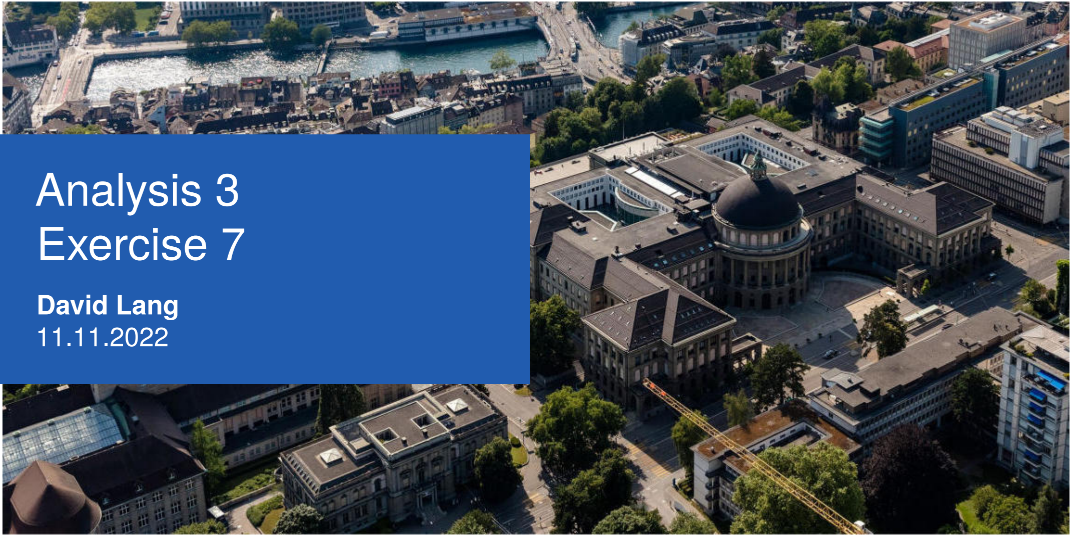

Analysis 3
Exercise 7

David Lang

11.11.2022

Outline

Outline

Exercise Evaluation

I kindly ask all of you to fill out the evaluation survey for my exercises.

You were invited via email and received a personalised link.

The answers are anonymous.

This should take a maximum of 10 minutes.

You will have the time to do it now.

I hope to receive honest opinions and suggestions.

This will help me a lot, thank you!

Outline

1. Exercise Evaluation
2. Serie 6 Review
3. Course Overview
4. Wave Equation
5. Domain of dependence and regions of influence
6. Nonhomogeneous Wave Equation
7. Tips for Serie 7

Serie 6 Review

1. Wave equation
2. Wave equation's anatomy
   - (a) $F = G$ means $F(.) = G(.)$
   - (b)

$$u(x,t) = \sum_{n=0}^{N} a_n \frac{\cos(n(x+ct)) + \cos(n(x-ct))}{2} + \frac{1}{2c} \sum_{n=0}^{M} b_n \int_{x-ct}^{x+ct} \cos(ny)\, dy$$
$$= \sum_{n=0}^{N} a_n \cos(nct)\cos(nx) + \sum_{n=1}^{M} \frac{b_n}{cn} \sin(nct)\cos(nx) + b_0 t$$

3. Propagation of symmetries from initial data
   - Periodic and even –> can be written as a sum of cosines.
4. Multiple choice
5. Time reversible
6. Zero boundary condition

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - **Wave equation**
    - **D'Alembert formula**
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

Wave Equation

Homogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy$$

Nonhomogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy$$

$$+ \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\, d\xi\, d\tau$$

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Domain of dependence and regions of influence

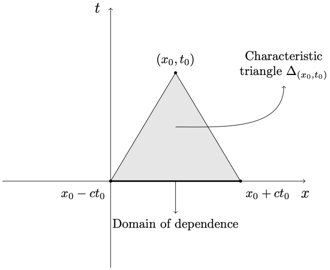

Figure 4.3: The characteristic triangle.

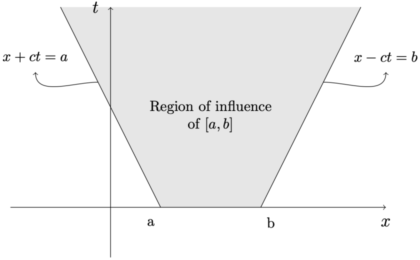

Figure 4.4: The region of influence.

Outline

Nonhomogeneous Wave Equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy + \frac{1}{2c} \int\int_{\Delta} F(\xi,\tau)\, d\xi\, d\tau$$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy + \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\, d\xi\, d\tau$$

$$v(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\, d\xi\, d\tau$$

$$\begin{cases} v_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

Example 1

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt & (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = cos(x) & x \in \mathbb{R} \end{cases}$$

Compute the explicit solution u.

Example 1

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt & (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = cos(x) & x \in \mathbb{R} \end{cases}$$

Compute the following limit:

$$\lim_{t \to \infty} \frac{u(x,t)}{t^3}$$

for every $x \in \mathbb{R}$.

Example 1

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt & (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = cos(x) & x \in \mathbb{R} \end{cases}$$

Compute $u(x,t) + u(x,-t)$ without using the explicit formula for u.

Particular Solution

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

particular solution $v(x,t)$
such that $v_{tt} - c^2 v_{xx} = F(x,t)$

$u = w + v$

Example 2

$$\begin{cases} u_{tt} - u_{xx} = 1 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = x^2 & x \in \mathbb{R} \\ u_t(x,0) = 1 & x \in \mathbb{R} \end{cases}$$

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Tips for Serie 7

1. (Non)homogeneous wave equation
   - (b) f and g are even, what does this imply?
2. Propagation of symmetries from initial data, II
   - (a) Define $v(x,t) = -u(-x,t)$ and then use the uniqueness theorem.
   - (b) Define $v(x,t) = u(x+L,t)$ and then use the uniqueness theorem.
3. Wave equation on a ring
   - Extend the boundary condition periodically, draw $x - x^2$ and the extension explicitly. Note we are only interested in one point: $u(1/2, 2022)$
4. Multiple choice
   - (a) What does d'Alembert's formula suggest?
   - (b) Same as (a)
   - (c) General solution of Exercise 6.2 (b)
5. Strange wave equation
   - Write down the PDE for $v(x,t) = \sqrt{u(x,t)}$, remember to use the chain rule.

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Before the next lecture:

1. 3Blue1Brown: But what is a partial differential equation?
   https://youtu.be/ly4S0oi3Yz8
2. 3Blue1Brown: Solving the Heat Equation
   https://youtu.be/ToIXSwZ1pJU

References:

1. Lecture notes on the course website.
2. "An Introduction to Partial Differential Equations" by Yehuda Pinchover and Jacob Rubinstein