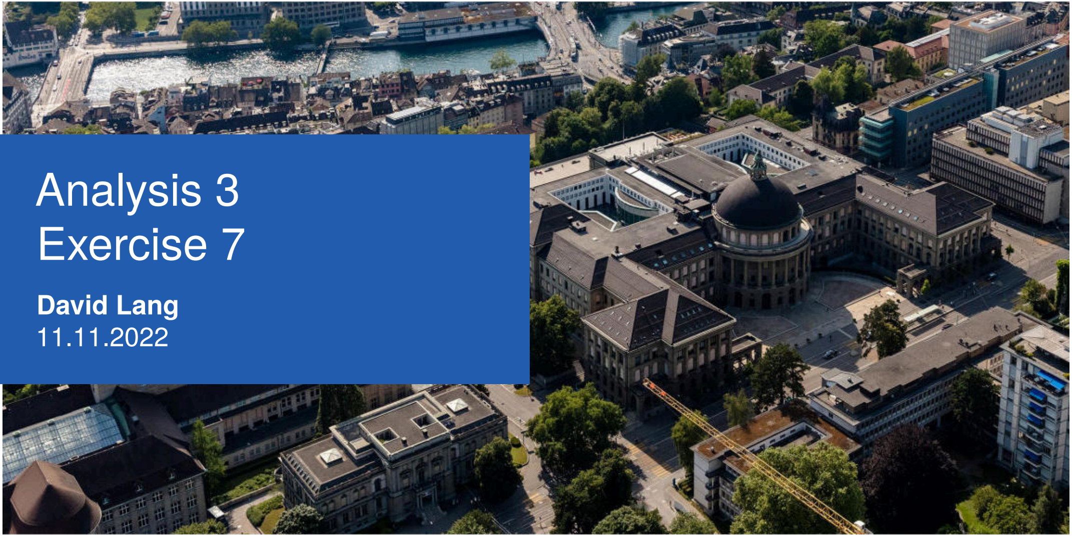

Analysis 3
Exercise 7

David Lang

11.11.2022

Outline

Outline

Exercise Evaluation

I kindly ask all of you to fill out the evaluation survey for my exercises.

You were invited via email and received a personalised link.

The answers are anonymous.

This should take a maximum of 10 minutes.

You will have the time to do it now.

I hope to receive honest opinions and suggestions.

This will help me a lot, thank you!

Outline

1. Exercise Evaluation
2. Serie 6 Review
3. Course Overview
4. Wave Equation
5. Domain of dependence and regions of influence
6. Nonhomogeneous Wave Equation
7. Tips for Serie 7

Serie 6 Review

1. Wave equation
2. Wave equation's anatomy
   - (a) $F = G$ means $F(.) = G(.)$ $F(\xi) = G(\xi) \Rightarrow f \equiv 0$
   - (b)

$$u(x,t) = \sum_{n=0}^{N} a_n \frac{\cos(n(x+ct)) + \cos(n(x-ct))}{2} + \frac{1}{2c} \sum_{n=0}^{M} b_n \int_{x-ct}^{x+ct} \cos(ny)\, dy$$

$$= \sum_{n=0}^{N} a_n \cos(nct)\cos(nx) + \sum_{n=1}^{M} \frac{b_n}{cn} \sin(nct)\cos(nx) + b_0 t$$

3. Propagation of symmetries from initial data
   - Periodic and even –> can be written as a sum of cosines.
4. Multiple choice
5. Time reversible
6. Zero boundary condition

Outline

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - **Wave equation**
    - **D'Alembert formula**
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

Wave Equation

Homogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

Nonhomogeneous

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau$$

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Domain of dependence and regions of influence

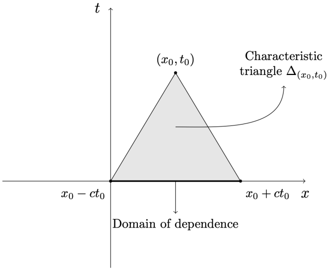

Figure 4.3: The characteristic triangle.

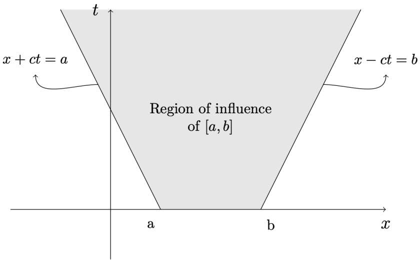

Figure 4.4: The region of influence.

Given a point,
↳ Find the domain on which it depends.

Given an interval,
↳ Find the region it influences.

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Nonhomogeneous Wave Equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c} \int\int_{\Delta} F(\xi,\tau)\,d\xi\,d\tau$$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau$$

$$v(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau$$

$$\begin{cases} v_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ v(x,0) = 0 & x \in \mathbb{R} \\ v_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

Example 1

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt & (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = cos(x) & x \in \mathbb{R} \end{cases}$$

$F(x,t) = xt$

$f(x) = 0$

$g(x) = \cos(x)$

Compute the explicit solution u.

$$u(x,t) = \frac{\cancel{f(x+ct)+f(x-ct)}}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi\,d\tau$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \xi\tau\,d\xi\,d\tau$$

$$= \frac{1}{2c}\Big[\sin(y)\Big]_{y=x-ct}^{y=x+ct} + \frac{1}{2c}\int_0^t \tau \Big[\tfrac{1}{2}\xi^2\Big]_{\xi=x-c(t-\tau)}^{\xi=x+c(t-\tau)} d\tau$$

$$= \frac{\sin(x+ct)-\sin(x-ct)}{2c} + \frac{1}{2c}\int_0^t \tau\cdot\Big(\tfrac{1}{2}(x+c(t-\tau))^2 - \tfrac{1}{2}(x-c(t-\tau))^2\Big)\,d\tau$$

Example 1

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt & (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = cos(x) & x \in \mathbb{R} \end{cases}$$

Compute the following limit:

$$\lim_{t \to \infty} \frac{u(x,t)}{t^3}$$

for every $x \in \mathbb{R}$.

$$= \frac{\sin(x+ct) - \sin(x-ct)}{2c} + \frac{1}{2c} \int_0^t \tau \cdot 2x \cdot c(t-\tau)\, d\tau$$

$$= \frac{\sin(x+ct) - \sin(x-ct)}{2c} + x \int_0^t \tau(t-\tau)\, d\tau$$

$$= \frac{\sin(x+ct) - \sin(x-ct)}{2c} + x \left[\frac{1}{2}\tau^2 t - \frac{1}{3}\tau^3\right]_{\tau=0}^{\tau=t}$$

$$= \frac{\sin(x+ct) - \sin(x-ct)}{2c} + x\left(\frac{1}{2}t^3 - \frac{1}{3}t^3\right)$$

$$= \frac{\sin(x+ct) - \sin(x-ct)}{2c} + \frac{xt^3}{6}$$

$$\lim_{t \to \infty} \frac{u(x,t)}{t^3}$$

$$= \lim_{t \to \infty} \frac{\sin(x+ct) - \sin(x-ct)}{2ct^3} + \frac{x}{6}$$

$$= \frac{x}{6}$$

Example 1

$$\begin{cases} u_{tt} - c^2 u_{xx} = xt & (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(x,0) = 0 & x \in \mathbb{R} \\ u_t(x,0) = cos(x) & x \in \mathbb{R} \end{cases}$$

Compute $u(x,t) + u(x,-t)$ without using the explicit formula for u.

define $v(x,t) = -u(x,-t)$

$v_{tt} = -u_{tt}(x,-t)\cdot(-1)\cdot(-1) = -u_{tt}(x,-t)$

$v_{xx} = -u_{xx}(x,-t)$

$v_{tt} - c^2 v_{xx}$

$= -u_{tt}(x,-t) + c^2 u_{xx}(x,-t)$

$= -\left(u_{tt}(x,-t) - c^2 u_{xx}(x,-t)\right)$

$= -\left(x\cdot(-t)\right) = xt$

$v(x,0) = -u(x,-0) = 0$

$v_t(x,0) = -u_t(x,-0)\cdot(-1) = u_t(x,0) = \cos(x)$

$$\begin{cases} v_{tt} - c^2 v_{xx} = xt \\ v(x,0) = 0 \\ v_t(x,0) = \cos(x) \end{cases}$$

By the uniqueness theorem for wave equations, $v(x,t) = u(x,t)$, i.e. $-u(x,-t) = u(x,t)$

$\Rightarrow u(x,t) + u(x,-t) = -u(x,-t) + u(x,-t)$

$\equiv 0$

Particular Solution

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

particular solution $v(x,t)$
such that $v_{tt} - c^2 v_{xx} = F(x,t)$

$u = w + v$

$w = u - v$

$w_{tt} - c^2 w_{xx}$

$= u_{tt} - v_{tt} - c^2(u_{xx} - v_{xx})$

$= \underbrace{(u_{tt} - c^2 u_{xx})}_{F(x,t)} - \underbrace{(v_{tt} - c^2 v_{xx})}_{F(x,t)}$

$= 0$

$w(x,0) = u(x,0) - v(x,0) = f(x) - v(x,0)$

$w_t(x,0) = u_t(x,0) - v_t(x,0) = g(x) - v_t(x,0)$

Thus, $w(x,t)$ solves the homogeneous wave equation with modified I.C.

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = f(x) - v(x,0) \\ w_t(x,0) = g(x) - v_t(x,0) \end{cases}$$

Example 2

$$\begin{cases} u_{tt} - u_{xx} = 1 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = x^2 & x \in \mathbb{R} \\ u_t(x,0) = 1 & x \in \mathbb{R} \end{cases}$$

$V_{tt}(t) = 1$

$V(t) = \frac{1}{2}t^2$) choose the simplest one

$w = u - V$

$$\begin{cases} w_{tt} - w_{xx} = 0 \\ w(x,0) = f(x) - V(0) = x^2 - \frac{1}{2}0^2 = x^2 \\ w_t(x,0) = g(x) - V_t(0) = 1 - 0 = 1 \end{cases}$$

$$w(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

$$= \frac{(x-t)^2 + (x+t)^2}{2} + \frac{1}{2}\int_{x-t}^{x+t} 1\,dy$$

$$= x^2 + t^2 + \frac{1}{2}\cdot 2t$$

$$= x^2 + t^2 + t$$

$$u(x,t) = w(x,t) + V(x,t)$$

$$= x^2 + t^2 + t + \frac{1}{2}t^2$$

$$= x^2 + \frac{3}{2}t^2 + t$$

Outline

Tips for Serie 7

1. (Non)homogeneous wave equation
   - (b) f and g are even, what does this imply?
2. Propagation of symmetries from initial data, II
   - (a) Define $v(x,t) = -u(-x,t)$ and then use the uniqueness theorem.
   - (b) Define $v(x,t) = u(x+L,t)$ and then use the uniqueness theorem.
3. Wave equation on a ring
   - Extend the boundary condition periodically, draw $x - x^2$ and the extension explicitly. Note we are only interested in one point: $u(1/2, 2022)$
4. Multiple choice
   - (a) What does d'Alembert's formula suggest?
   - (b) Same as (a)
   - (c) General solution of Exercise 6.2 (b)
5. Strange wave equation
   - Write down the PDE for $v(x,t) = \sqrt{u(x,t)}$, remember to use the chain rule.

Before the next lecture:

1. 3Blue1Brown: But what is a partial differential equation?
   https://youtu.be/ly4S0oi3Yz8
2. 3Blue1Brown: Solving the Heat Equation
   https://youtu.be/ToIXSwZ1pJU

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein