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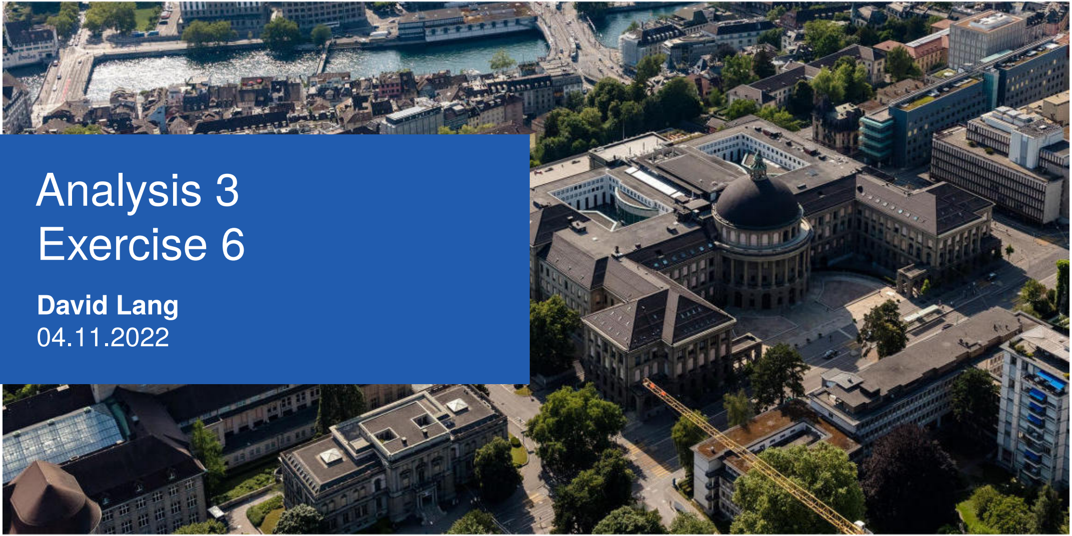

Analysis 3
Exercise 6

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04.11.2022

Outline

1. Serie 5 Review
2. Course Overview
3. Wave Equation
4. Canonical Form and Change of Variables
5. D'Alembert Formula
6. Examples
7. Tips for Serie 6

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Serie 5 Review

1. Weak solutions
   - Not differentiable –> no chain rule
2. Balance laws
   - Characteristics are not straight lines anymore.
3. Multiple choice
   - An antisymmetric matrix must have zeros on its diagonal.
4. Weak solutions II
   - slope $1/c(u_0(s)) = 1/e^{-u_0(s)} = e^{u_0(s)}$
5. Finding shock waves
   - $\frac{d}{dy}\gamma(y) = \gamma_y(y) = \gamma'(y)$

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws

- 2nd order PDEs
  - Hyperbolic PDEs
    - **Wave equation**
    - **D'Alembert formula**
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

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Wave Equation

The homogeneous wave equation in one (spatial) dimension has the form

$$u_{tt} - c^2 u_{xx} = 0, x \in \mathbb{R}, t > 0$$

$c \in \mathbb{R}$ is called the wave speed.

Note that $x \in \mathbb{R}$, which means that the problem can be thought of as the amplitude of the vibration of an **infinite** string.

This is the homogeneous wave equation, i.e. no external force.

If we impose boundary conditions (maybe only looking at $[0, L]$), then we will have to do some modifications, such as using the method of **Separation of Variables**.

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Canonical Form and Change of Variables

$$\xi(x,t) = x + ct \ \ \& \ \ \eta(x,t) = x - ct$$
$$u(x,t) = w(\xi, \eta)$$

$$u_t = w_\xi \xi_t + w_\eta \eta_t \ \ \& \ \ u_x = w_\xi \xi_x + w_\eta \eta_x$$
$$u_{tt} = c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}) \ \ \& \ \ u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$
$$u_{tt} - c^2 u_{xx} = 0 = -4c^2 w_{\xi\eta}$$

Canonical Form and Change of Variables

The result from the previous slide:

$$\frac{\partial}{\partial \eta} w_\xi = 0$$

w_ξ is independent of η

$$w_\xi(\xi, \eta) = f(\xi)$$

Integrate with respect to ξ, we get

$$w(\xi, \eta) = F(\xi) + G(\eta)$$

Transform back to the original coordinates

$$u(x, t) = F(x + ct) + G(x - ct)$$

Characteristics of the Wave Equation

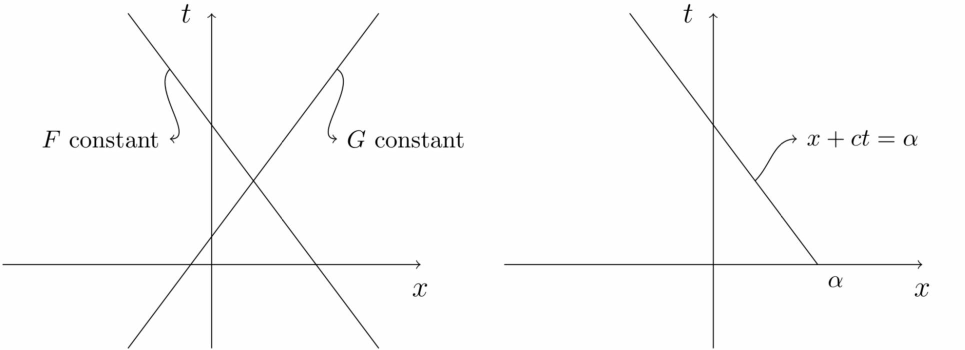
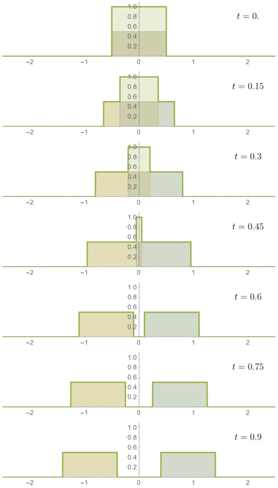
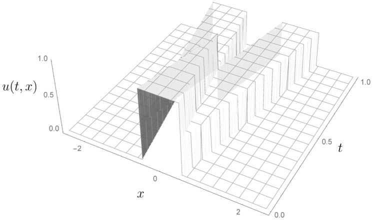
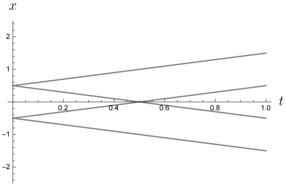


Figure 4.1: On the left, the characteristics where F and G are constant. On the right, the backward wave $F(x + ct)$.

Characteristics of the Wave Equation







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D'Alembert Formula

Cauchy problem for the wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

We want $u(x,t)$ to have the form $F(x+ct) + G(x-ct)$.

By plugging in the values for $t = 0$, we could find out what F and G are and thus the general solution.

D'Alembert Formula

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy$$

Note: the value of the solution at (x_0, t_0) is only influenced by the values of f and g in $[x_0 - ct_0, x_0 + ct_0]$

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Example 1

Let $u(x,t)$ be the solution of the following initial value problem

$$\begin{cases} u_{tt} = 4u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$f(x) = \begin{cases} 3 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

$$g(x) = \begin{cases} 1 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

Find $u(1,1)$

Example 1

Let $u(x,t)$ be the solution of the following initial value problem

$$\begin{cases} u_{tt} = 4u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$f(x) = \begin{cases} 3 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

$$g(x) = \begin{cases} 1 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

Find $\lim_{t\to\infty} u(1,t)$

Example 2

Consider the initial value problem with zero boundary condition

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(x,0) = x^4 & x \in [0,\infty) \\ u_t(x,0) = sin(x) & x \in [0,\infty) \end{cases}$$

Evaluate $u(2,1)$ and $u(1,2)$. In which of the two points $((2,1)\ or\ (1,2))$ is the solution unaffected by the boundary condition at $x = 0$?

Example 2

Consider the initial value problem with zero boundary condition

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(x,0) = x^4 & x \in [0,\infty) \\ u_t(x,0) = sin(x) & x \in [0,\infty) \end{cases}$$

The D'Alembrt formula is derived for $x \in \mathbb{R}$, we need to modify the problem before applying it.

We should define a new problem on $\mathbb{R}$ so that its solution provides the correct result if we just focus on $x \geq 0$.

Example 2

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(x,0) = x^4 & x \in [0,\infty) \\ u_t(x,0) = sin(x) & x \in [0,\infty) \end{cases}$$

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Tips for Serie 6

1. Wave equation
   - D'Alembert's Formula
     $sin(a+b) + sin(a-b) = 2sin(a)cos(b)$ & $cos(a+b) + cos(a-b) = 2cos(a)cos(b)$
2. Wave equation's anatomy
   - (a) Chapter 4.2 in Script: The Cauchy problem and d'Alembert's formula
   - (b) Apply d'Alembert's formula directly
     $sin(a+b) + sin(a-b) = 2sin(a)cos(b)$ & $cos(a+b) + cos(a-b) = 2cos(a)cos(b)$
3. Propagation of symmetries from initial data
   - (a) Chapter 4.5 in Script: Symmetry of the wave equation
   - (b) Periodic –> Fourier series, even –> Which terms of the Fourier Series disappear?
4. Multiple choice
   -
5. Time reversible
   - Check the properties one by one.
6. Zero boundary condition
   - Example 2 of today's exercise.

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Before the next lecture:

1. 3Blue1Brown: But what is a partial differential equation?
   https://youtu.be/ly4S0oi3Yz8
2. 3Blue1Brown: Solving the Heat Equation
   https://youtu.be/ToIXSwZ1pJU

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. NDSU lecture notes