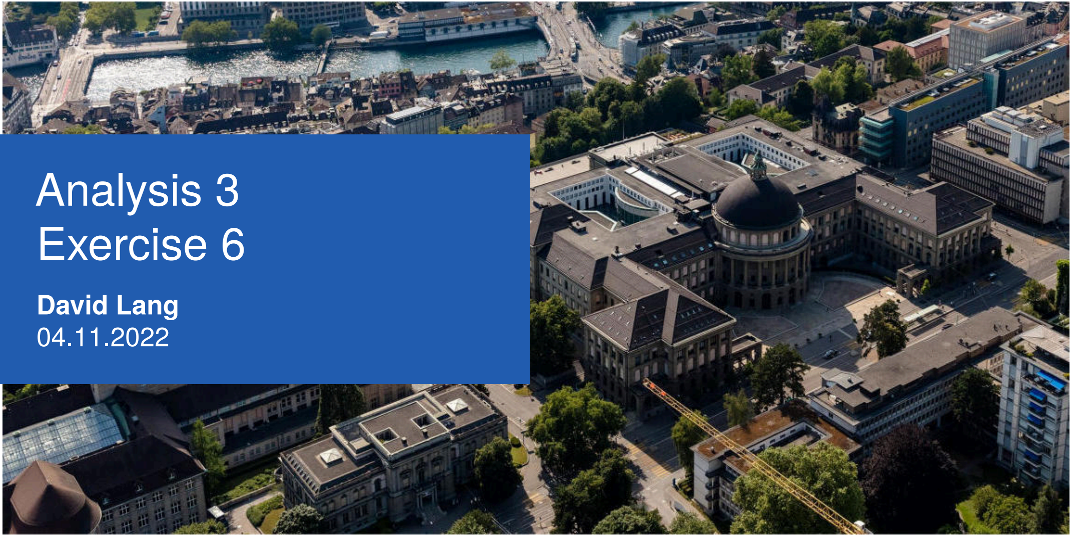


Analysis 3
Exercise 6

David Lang

04.11.2022

Outline

Outline

Serie 5 Review

1. Weak solutions
   - Not differentiable –> no chain rule
2. Balance laws
   - Characteristics are not straight lines anymore.
3. Multiple choice
   - An antisymmetric matrix must have zeros on its diagonal. $A = -A^T$
4. Weak solutions II
   - slope $1/c(u_0(s)) = 1/e^{-u_0(s)} = e^{u_0(s)}$
5. Finding shock waves
   - $\frac{d}{dy}\gamma(y) = \gamma_y(y) = \gamma'(y)$

$x = \gamma(y)$

$\begin{cases} u(x,y) = \dots & x < \gamma(y) \\ u(x,y) = \dots & x > \gamma(y) \end{cases}$

Outline

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - **Wave equation**
    - **D'Alembert formula**
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

Wave Equation

The homogeneous wave equation in one (spatial) dimension has the form

$$u_{tt} - c^2 u_{xx} = 0, x \in \mathbb{R}, t > 0$$

$c \in \mathbb{R}$ is called the wave speed.

Note that $x \in \mathbb{R}$, which means that the problem can be thought of as the amplitude of the vibration of an **infinite** string.

This is the homogeneous wave equation, i.e. no external force.

If we impose boundary conditions (maybe only looking at $[0, L]$), then we will have to do some modifications, such as using the method of **Separation of Variables**.

$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g$

$a = -c^2 \quad c = 1$

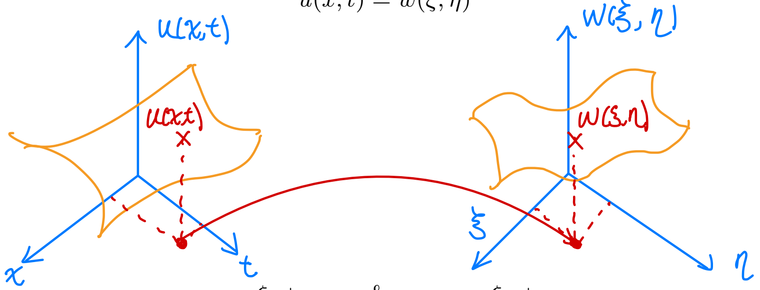
$\delta(L) = b^2 - ac = c^2 > 0$ hyperbolic

Outline

Canonical Form and Change of Variables

$$\xi(x,t) = x + ct \quad \& \quad \eta(x,t) = x - ct$$

$$u(x,t) = w(\xi,\eta)$$



$$u_t = w_\xi \xi_t + w_\eta \eta_t \quad \& \quad u_x = w_\xi \xi_x + w_\eta \eta_x$$

$$u_{tt} = c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}) \quad \& \quad u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$

$$u_{tt} - c^2 u_{xx} = 0 = -4c^2 w_{\xi\eta}$$

Canonical Form and Change of Variables

The result from the previous slide:

$$\frac{\partial}{\partial \eta} w_\xi = 0$$

w_ξ is independent of η

$$w_\xi(\xi, \eta) = f(\xi)$$

Integrate with respect to ξ, we get

$$w(\xi, \eta) = F(\xi) + G(\eta)$$

Transform back to the original coordinates

$$u(x, t) = F(x + ct) + G(x - ct)$$

backward forward

Characteristics of the Wave Equation

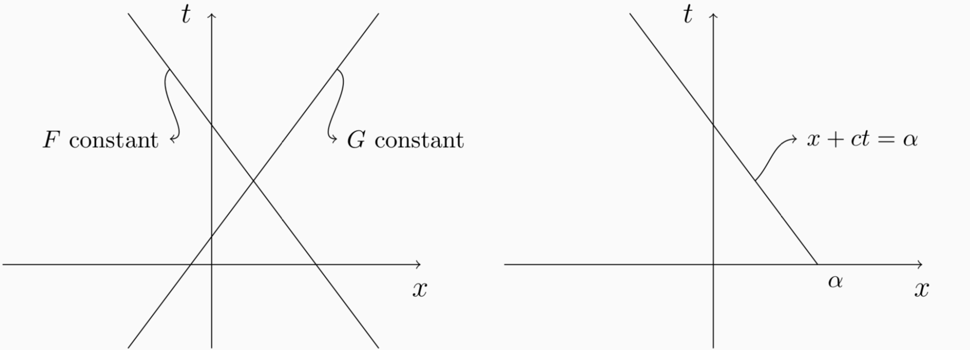
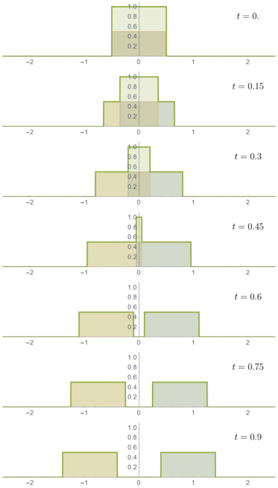
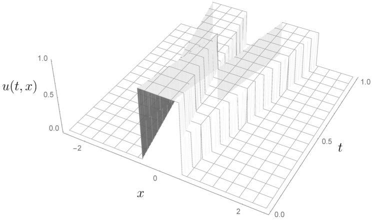
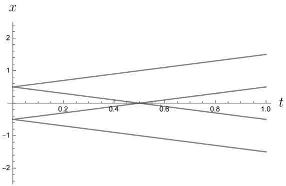


Figure 4.1: On the left, the characteristics where F and G are constant. On the right, the backward wave $F(x + ct)$.

Characteristics of the Wave Equation







Outline

D'Alembert Formula

Cauchy problem for the wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

We want $u(x,t)$ to have the form $F(x+ct) + G(x-ct)$.

By plugging in the values for $t = 0$, we could find out what F and G are and thus the general solution.

D'Alembert Formula

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy$$

Note: the value of the solution at (x_0, t_0) is only influenced by the values of f and g in $[x_0 - ct_0, x_0 + ct_0]$

Outline

1. Serie 5 Review
2. Course Overview
3. Wave Equation
4. Canonical Form and Change of Variables
5. D'Alembert Formula
6. Examples
7. Tips for Serie 6

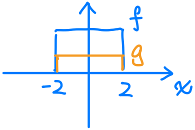
Example 1

Let $u(x,t)$ be the solution of the following initial value problem

$$\begin{cases} u_{tt} = 4u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$f(x) = \begin{cases} 3 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

$$g(x) = \begin{cases} 1 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$



Find $u(1,1)$

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

$c = 2$

$$u(1,1) = \frac{f(1+2)+f(1-2)}{2} + \frac{1}{2\cdot 2}\int_{1-2}^{1+2} g(y)\,dy$$

$$= \frac{f(3)+f(-1)}{2} + \frac{1}{4}\int_{-1}^{3} 1\,dy$$

$$= \frac{0+3}{2} + \frac{1}{4}\int_{-1}^{2} 1\,dy$$

$$= \frac{3}{2} + \frac{1}{4}\cdot 3$$

$$= \frac{9}{4}$$

Example 1

Let $u(x,t)$ be the solution of the following initial value problem

$$\begin{cases} u_{tt} = 4u_{xx} & x \in \mathbb{R}, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$f(x) = \begin{cases} 3 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

$$g(x) = \begin{cases} 1 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

Find $\lim_{t\to\infty} u(1,t)$

$$\lim_{t\to\infty} u(1,t) = \lim_{t\to\infty} \frac{1}{2}[f(1+2t)+f(1-2t)] + \frac{1}{4}\int_{1-2t}^{1+2t} g(y)\,dy$$

$$= 0 + \frac{1}{4}\int_{-2}^{2} 1\,dy$$

$$= 0 + \frac{1}{4}\cdot 4$$

$$= 1$$

Example 2

Consider the initial value problem with zero boundary condition

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(x,0) = x^4 & x \in [0,\infty) \\ u_t(x,0) = sin(x) & x \in [0,\infty) \end{cases}$$

Evaluate $u(2,1)$ and $u(1,2)$. In which of the two points $((2,1)\ or\ (1,2))$ is the solution unaffected by the boundary condition at $x = 0$?

If $v(x,t)$ is an odd function, then it automatically satisfies $v(0,t)=0$ for all t

Extend $u(x,0)$ & $u_t(x,0)$ as odd functions to all $\mathbb{R}$.

$$\begin{cases} v_{tt} - v_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ v(x,0) = x^3 \cdot |x| & x \in \mathbb{R} \\ v_t(x,0) = \sin(x) & x \in \mathbb{R} \end{cases}$$

Example 2

Consider the initial value problem with zero boundary condition

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(x,0) = x^4 & x \in [0,\infty) \\ u_t(x,0) = sin(x) & x \in [0,\infty) \end{cases}$$

The D'Alembrt formula is derived for $x \in \mathbb{R}$, we need to modify the problem before applying it.

We should define a new problem on $\mathbb{R}$ so that its solution provides the correct result if we just focus on $x \geq 0$.

Example 2

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in (0,\infty) \times (0,\infty) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(x,0) = x^4 & x \in [0,\infty) \\ u_t(x,0) = sin(x) & x \in [0,\infty) \end{cases}$$

$$\begin{cases} v_{tt} - v_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ v(x,0) = x^3 \cdot |x| & x \in \mathbb{R} \\ v_t(x,0) = \sin(x) & x \in \mathbb{R} \end{cases}$$

We get $u(x,t)$ by just restricting $v(x,t)$ to $x \geq 0$.

$$v(x,t) = \frac{1}{2}\left[(x+t)^3|x+t| + (x-t)^3|x-t|\right] + \frac{1}{2}\int_{x-t}^{x+t} \sin(y)\,dy$$

$$= \frac{1}{2}\left[(x+t)^3|x+t| + (x-t)^3|x-t|\right] - \frac{1}{2}\left[\cos(x+t) - \cos(x-t)\right]$$

$v(1,2) = \frac{1}{2}\left(3^4 - 1^4\right) + \frac{1}{2}\left(\cos(1) - \cos(3)\right) = 40 + \frac{1}{2}\left(\cos(1) - \cos(3)\right)$ $\quad u(1,2) = v(1,2)$ $\quad x = 1 > 0$

$v(2,1) = \frac{1}{2}\left(3^4 + 1^4\right) + \frac{1}{2}\left(\cos(1) - \cos(3)\right) = 41 + \frac{1}{2}\left(\cos(1) - \cos(3)\right)$ $\quad u(2,1) = v(2,1)$ $\quad x = 2 > 0$

domain of dependence:

$(1,2)$: $[x_0 - ct_0, x_0 + ct_0] = [-1, 3]$ → affected by the zero boundary condition

$(2,1)$: $[x_0 - ct_0, x_0 + ct_0] = [1, 3]$ → unaffected by the zero boundary condition

Outline

Tips for Serie 6

1. Wave equation
   - D'Alembert's Formula
     $sin(a+b) + sin(a-b) = 2sin(a)cos(b)$ & $cos(a+b) + cos(a-b) = 2cos(a)cos(b)$
2. Wave equation's anatomy
   - (a) Chapter 4.2 in Script: The Cauchy problem and d'Alembert's formula
   - (b) Apply d'Alembert's formula directly
     $sin(a+b) + sin(a-b) = 2sin(a)cos(b)$ & $cos(a+b) + cos(a-b) = 2cos(a)cos(b)$
3. Propagation of symmetries from initial data
   - (a) Chapter 4.5 in Script: Symmetry of the wave equation
   - (b) Periodic –> Fourier series, even –> Which terms of the Fourier Series disappear?
4. Multiple choice
   -
5. Time reversible
   - Check the properties one by one.
6. Zero boundary condition
   - Example 2 of today's exercise.

ETH zürich

Before the next lecture:

1. 3Blue1Brown: But what is a partial differential equation?
   https://youtu.be/ly4S0oi3Yz8
2. 3Blue1Brown: Solving the Heat Equation
   https://youtu.be/ToIXSwZ1pJU

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. NDSU lecture notes