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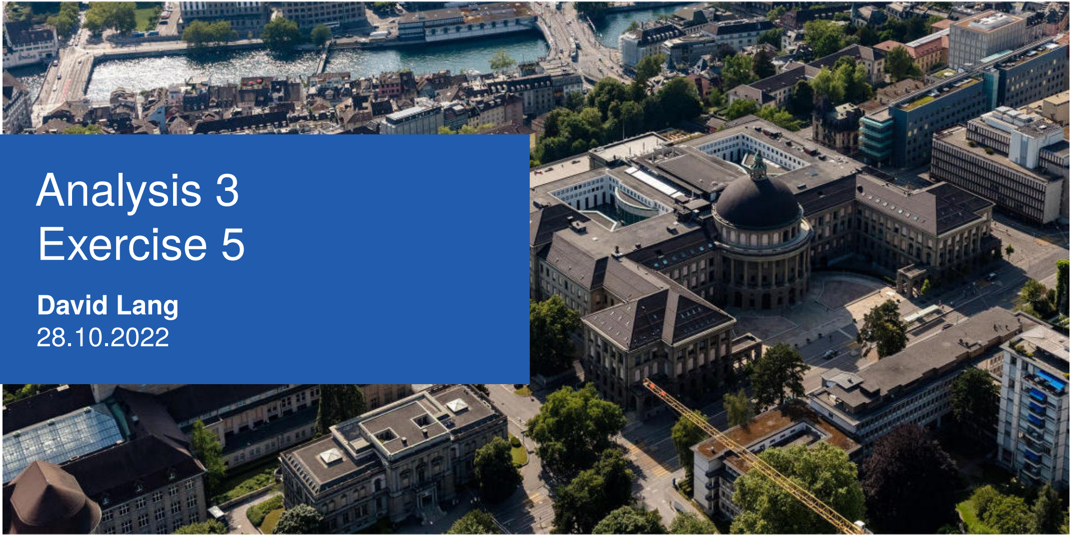

Analysis 3
Exercise 5

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Outline

Outline

Serie 4 Review

1. Conservation laws and critical times
   - Well done!
2. Multiple choice
   - (c) See today's example.
3. Weak solutions
   - (b)

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{0} f(x)\,dx + \int_{0}^{x_1} f(x)\,dx \quad \text{for } x_0 < 0 < x_1$$

Outline

1. Serie 4 Review
2. Course Overview
3. Rankine-Hugoniot Condition and Entropy condition
4. Examples
5. Classification of linear second order PDEs
6. Tips for Serie 5

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - **Conservation laws**
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

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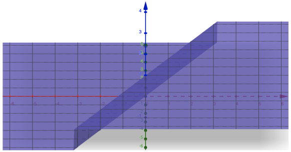
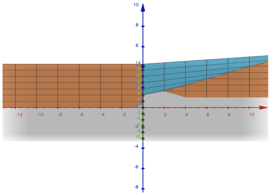
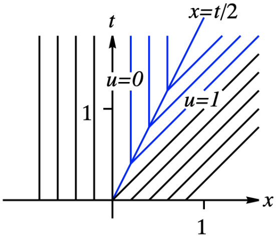
Rankine-Hugoniot Condition and Entropy condition

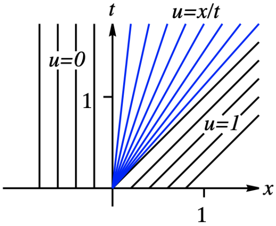
$$\begin{cases} u_y + uu_x = 0 \\ u(x, y = 0) = 0 \quad \text{for } x < 0 \\ u(x, y = 0) = 1 \quad \text{for } x > 0 \end{cases}$$

Rankine-Hugoniot Condition and Entropy condition









Entropy condition

The second law of thermodynamics: in a closed system information is only lost as time y increases, and cannot be created.

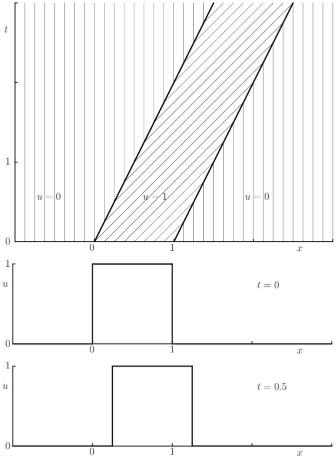
Characteristics carry with them information on the solution of a first-order PDE.

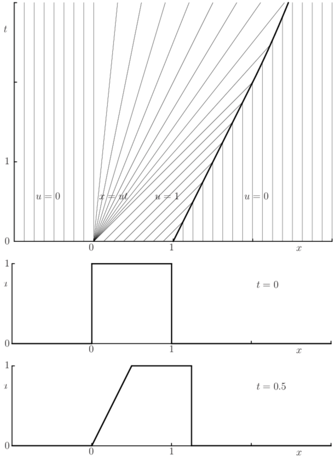
Therefore the emergence of a characteristic **from** a shock is interpreted as a **creation** of information which should be forbidden.

$$f'(u^+) < \gamma' < f'(u^-)$$
$$c(u^+) < \gamma' < c(u^-)$$

Special case for Burgers' Equation, the shock solution is only valid if $u^- > u^+$.

Graphical Illustration





Outline

Example 1

$$\begin{cases} u_y + u^2 u_x = 0 & \\ u(x, y = 0) = 1 & \text{for } x \leq 0 \\ u(x, y = 0) = \sqrt{1 - x} & \text{for } 0 < x < 1 \\ u(x, y = 0) = 0 & \text{for } x \geq 1 \end{cases}$$

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\begin{cases}
u_y + u^2 u_x = 0 & \\
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u(x, y = 0) = \sqrt{1 - x} & \text{for } 0 < x < 1 \\
u(x, y = 0) = 0 & \text{for } x \geq 1
\end{cases}
$$

Example 2

$$\begin{cases} u_y + u^2 u_x = 0 \\ u(x, y=0) = 1 \quad \text{for } x < 0 \\ u(x, y=0) = 3 \quad \text{for } x > 0 \end{cases}$$

Determine the characteristics inside the regions:

$x < y$ and $x > 9y$

Example 2

$$
\begin{cases} u_y + u^2 u_x = 0 \\ u(x, y=0) = 1 & \text{for } x < 0 \\ u(x, y=0) = 3 & \text{for } x > 0 \end{cases}
$$

Verify that the following are weak solutions:

$$
u_1(x,y) = \begin{cases} 1 & \text{for } x < \frac{7}{3}y \\ 2 & \text{for } \frac{7}{3}y < x < 4y \\ \sqrt{\frac{x}{y}} & \text{for } 4y < x < 9y \\ 3 & \text{for } x > 9y \end{cases}
$$

$$
u_2(x,y) = \begin{cases} 1 & \text{for } x < y \\ \sqrt{\frac{x}{y}} & \text{for } y < x < 9y \\ 3 & \text{for } x > 9y \end{cases}
$$

Example 2

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Which solution satisfies the entropy condition?

Outline

Frame Title

A General linear second-order PDE in two independent variables has the form:

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

The discriminant:

$$\delta(L)(x_0, y_0) := b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0)$$

$\delta(L)(x_0, y_0) > 0$: hyperbolic

$\delta(L)(x_0, y_0) = 0$: parabolic

$\delta(L)(x_0, y_0) < 0$: elliptic

Outline

Tips for Serie 5

1. Weak solutions
   - (a) Remember the chain rule.
   - (b) Check the Rankie-Hugoniot condition.
     What did we do wrong? When can we apply the chain rule?
2. Balance laws
   - (a) Write $u(x, y)$ as some terms plus the integral of g.
   - (b) Check the form of the characteristics.
3. Multiple choice
   - Check the definition in the script.
4. Weak solutions II
   - (b) Check the two inequalities of the entropy condition separately.
5. Finding shock waves
   - Done in the exercise session.

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. “Analysis in Mechanical Engineering” by Leon van Dommelen
4. Standford math220a handouts