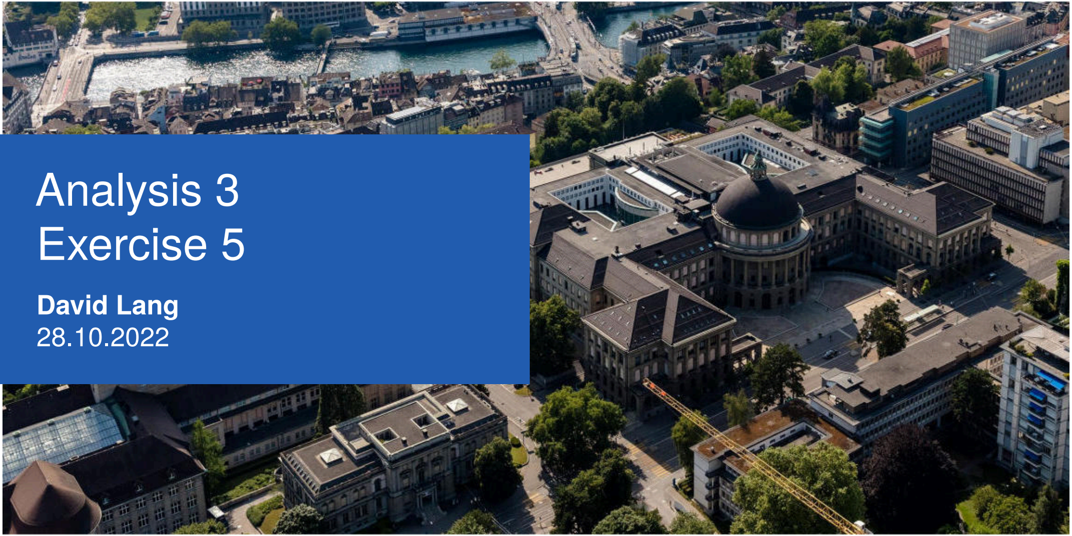
D-MATH
Analysis 3



Analysis 3
Exercise 5

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28.10.2022

Outline

Outline

Serie 4 Review

1. Conservation laws and critical times
   - Well done!
2. Multiple choice
   - (c) See today's example.
3. Weak solutions
   - (b)

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{0} f(x)\,dx + \int_{0}^{x_1} f(x)\,dx \quad \text{for } x_0 < 0 < x_1$$

Outline

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - **Conservation laws**
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

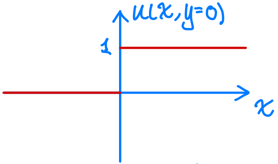
1. Serie 4 Review
2. Course Overview
3. Rankine-Hugoniot Condition and Entropy condition
4. Examples
5. Classification of linear second order PDEs
6. Tips for Serie 5

Rankine-Hugoniot Condition and Entropy condition

$c(u) = u$

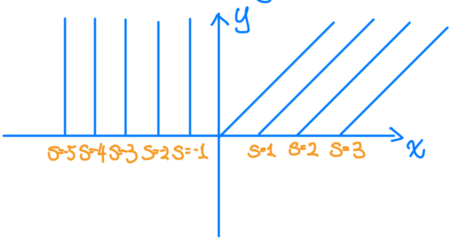
$f(u) = \frac{1}{2}u^2$

$$\begin{cases} u_y + u u_x = 0 \\ u(x, y=0) = 0 \quad \text{for } x < 0 \\ u(x, y=0) = 1 \quad \text{for } x > 0 \end{cases}$$



u should be constant along the projected characteristic curves : $x = h(s)y + s$

$$\begin{cases} s<0 & h(s)=0 & x=s \\ s>0 & h(s)=1 & x=y+s \end{cases}$$



$$u_1(x,y) = \begin{cases} 0 & x < y/2 \\ 1 & x > y/2 \end{cases}$$

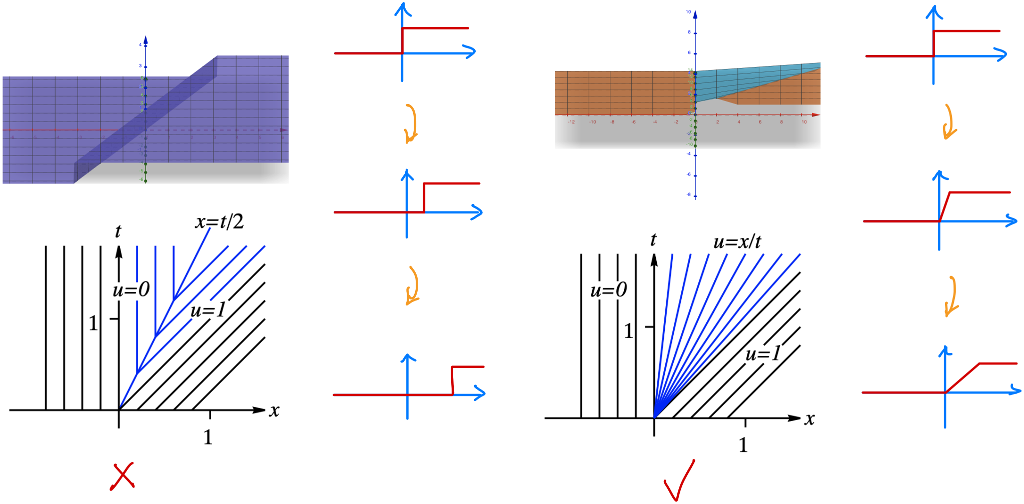
$u_1(x,y)$ satisfies the Rankine Hugoniot condition

$$u_2(x,y) = \begin{cases} 0 & x \leq 0 \\ x/y & 0 < x < y \\ 1 & x \geq y \end{cases}$$

$u_2(x,y)$ is a continuous solution : rarefaction wave

? which one should we choose ?

Rankine-Hugoniot Condition and Entropy condition



Entropy condition

The second law of thermodynamics: in a closed system information is only lost as time y increases, and cannot be created.

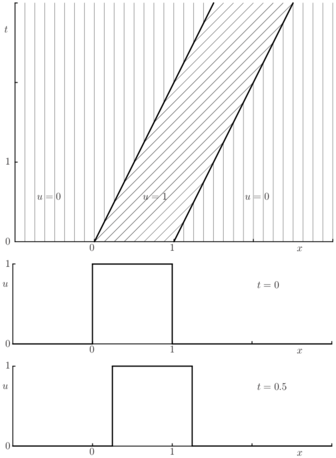
Characteristics carry with them information on the solution of a first-order PDE.

Therefore the emergence of a characteristic **from** a shock is interpreted as a **creation** of information which should be forbidden.

$$f'(u^+) < \gamma' < f'(u^-)$$
$$c(u^+) < \gamma' < c(u^-)$$

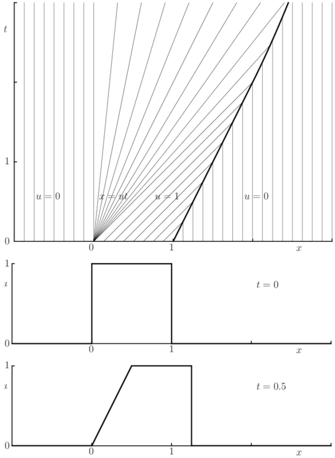
Special case for Burgers' Equation, the shock solution is only valid if $u^- > u^+$.

Graphical Illustration



t increases

entropy condition not fulfilled ✗



t increases

entropy condition fulfilled ✓

Outline

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4. Examples
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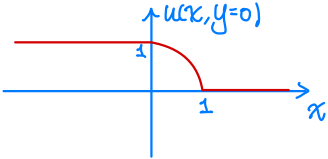
6. Tips for Serie 5

Example 1

$$\begin{cases} u_y + u^2 u_x = 0 & \\ u(x, y=0) = 1 & \text{for } x \le 0 \\ u(x, y=0) = \sqrt{1-x} & \text{for } 0 < x < 1 \\ u(x, y=0) = 0 & \text{for } x \ge 1 \end{cases}$$



$f(u) = \frac{1}{3}u^3 \quad c(u) = u^2 \quad c'(u) = 2u$

$$T(s) = \begin{pmatrix} s \\ 0 \\ \hat{u}_0(s) \end{pmatrix} \qquad \hat{u}_0(s) = \begin{cases} 1 & s \le 0 \\ \sqrt{1-s} & 0 < s < 1 \\ 0 & s \ge 1 \end{cases}$$

To compute y_c

$$c'(u_0(s)) \cdot u_0'(s) = \begin{cases} 2 \cdot 1 \cdot 0 = 0 & s \le 0 \\ 2 \cdot \sqrt{1-s} \cdot \frac{-1}{2\sqrt{1-s}} = -1 & 0 < s < 1 \\ 2 \cdot 0 \cdot 0 = 0 & s \ge 1 \end{cases}$$

$$y_c = \inf_{s \in \mathbb{R}:\, c'(u_0(s))\, u_0'(s) < 0} \left\{ \frac{-1}{c'(u_0(s))\, u_0'(s)} \right\} = 1$$

Example 1

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$$\begin{cases} x_t = \hat{u}^2 & x(t=0, s) = s \\ y_t = 1 & y(t=0, s) = 0 \\ \hat{u}_t = 0 & \hat{u}(t=0, s) = \hat{u}_0(s) \end{cases}$$

Last solve: $x(t,s) = \hat{u}_0(s)^2 \cdot t + s$

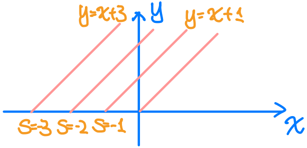
First solve: $\boxed{y(t,s) = t}$

Then solve: $\hat{u}(t,s) = \hat{u}_0(s)$

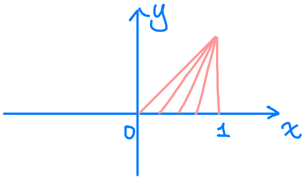
$\rightarrow$

$$\begin{cases} s \leq 0 & x = 1^2 \cdot y + s \Rightarrow s = x - y, \; t = y \\ 0 < s < 1 & x = (1-s)y + s \Rightarrow s = \dfrac{x-y}{1-y}, \; t = y \\ s \geq 1 & x = 0 \cdot y + s \Rightarrow s = x, \; t = y \end{cases}$$

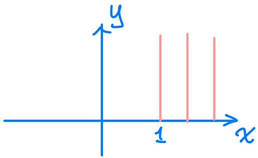
$s \leq 0 \quad y = x - s$



$0 < s < 1 \quad y = \dfrac{1}{1-s}x - \dfrac{s}{1-s}$

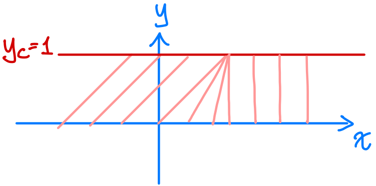


$s > 1 \quad x = s$



Example 1

$$\begin{cases} u_y + u^2 u_x = 0 & \\ u(x, y=0) = 1 & \text{for } x \le 0 \\ u(x, y=0) = \sqrt{1-x} & \text{for } 0 < x < 1 \\ u(x, y=0) = 0 & \text{for } x \ge 1 \end{cases}$$



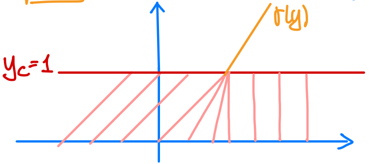
$y \le 1$

$$u(x,y) = \begin{cases} 1 & x \le y \\ \sqrt{\frac{1-x}{1-y}} & y < x < 1 \\ 0 & x \ge 1 \end{cases}$$

We look for a weak solution with a shock wave, starting from time $y > 1$

The shock wave will start at $(1,1)$ and so we look for a solution of the form

$$u(x,y) = \begin{cases} 1 & x < \gamma(y) \\ 0 & x > \gamma(y) \end{cases}$$



$y > 1$

$$u(x,y) = \begin{cases} 1 & x < \gamma(y) \\ 0 & x > \gamma(y) \end{cases}$$

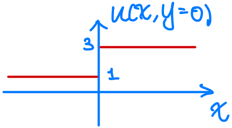
To find $\gamma(y)$, we apply the Rankie Hugoniot condition

$$\gamma'(y) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{\frac{1}{3} 0^3 - \frac{1}{3} \cdot 1^3}{0 - 1} = \frac{1}{3}$$

$$\boxed{\gamma(y) = \frac{1}{3} y + \frac{2}{3}}$$

Example 2

$$\begin{cases} u_y + u^2 u_x = 0 \\ u(x, y=0) = 1 \quad \text{for } x < 0 \\ u(x, y=0) = 3 \quad \text{for } x > 0 \end{cases}$$



$$\begin{cases} x_t = \hat{u}^2 & x(t=0,s)=s \\ y_t = 1 & y(t=0,s)=0 \\ \hat{u}_t = 0 & \hat{u}(t=0,s)=\hat{u}_0(s) \end{cases} \qquad \hat{u}_0(s) = \begin{cases} 1 & s<0 \\ 3 & s>0 \end{cases}$$

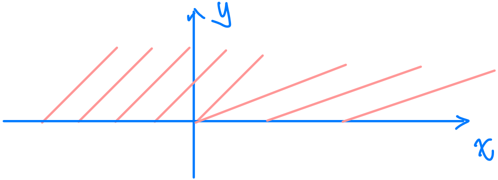
Determine the characteristics inside the regions:

$x < y$ and $x > 9y$

$$\begin{cases} \text{Last solve: } x(t,s) = \hat{u}_0(s)^2 t + s \\ \text{First solve: } y(t,s) = t \\ \text{Then solve: } \hat{u}(t,s) = \hat{u}_0(s) \end{cases}$$

$$\begin{cases} x = y + s & \text{for } s<0 \\ x = 9y + s & \text{for } s>0 \end{cases} \Rightarrow \begin{cases} t=y,\ s=x-y & \text{for } x<y \\ t=y,\ s=x-9y & \text{for } x>9y \end{cases}$$

$$\begin{cases} u(x,y) = 1 & x<y \\ u(x,y)=3 & x>9y \end{cases}$$



Example 2

$$\begin{cases} u_y + u^2 u_x = 0 \\ u(x, y=0) = 1 & \text{for } x < 0 \\ u(x, y=0) = 3 & \text{for } x > 0 \end{cases}$$

Verify that the following are weak solutions:

$$u_1(x,y) = \begin{cases} 1 & \text{for } x < \frac{7}{3}y \\ 2 & \text{for } \frac{7}{3}y < x < 4y \\ \sqrt{\frac{x}{y}} & \text{for } 4y < x < 9y \\ 3 & \text{for } x > 9y \end{cases}$$

$$u_2(x,y) = \begin{cases} 1 & \text{for } x < y \\ \sqrt{\frac{x}{y}} & \text{for } y < x < 9y \\ 3 & \text{for } x > 9y \end{cases}$$

To show that u_1 is a weak solution, we need to first identify where (if any) discontinuities are: $\{x = \frac{7}{3}y\}$

I. In each region, u_1 satisfies the PDE

II. Across $x = \frac{7}{3}y$, u_1 satisfies the Rankie Hugoniot condition

I. constants ✓

$$\frac{\partial}{\partial y}\sqrt{\frac{x}{y}} + \left(\frac{x}{y}\right)\frac{\partial}{\partial x}\sqrt{\frac{x}{y}} = 0 \checkmark$$

II. $$\frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{\frac{1}{3}2^3 - \frac{1}{3}1^3}{2-1} = \frac{7}{3} \checkmark$$

$\Rightarrow$ u_1 is a weak solution

Example 2

$$\begin{cases} u_y + u^2 u_x = 0 \\ u(x, y=0) = 1 & \text{for } x < 0 \\ u(x, y=0) = 3 & \text{for } x > 0 \end{cases}$$

Verify that the following are weak solutions:

$$u_1(x,y) = \begin{cases} 1 & \text{for } x < \frac{7}{3}y \\ 2 & \text{for } \frac{7}{3}y < x < 4y \\ \sqrt{\frac{x}{y}} & \text{for } 4y < x < 9y \\ 3 & \text{for } x > 9y \end{cases}$$

$$u_2(x,y) = \begin{cases} 1 & \text{for } x < y \\ \sqrt{\frac{x}{y}} & \text{for } y < x < 9y \\ 3 & \text{for } x > 9y \end{cases}$$

Which solution satisfies the entropy condition?

u_2 is continuous everywhere, so we just need to check that each part of u_2 satisfies the PDE, same as what we did for u_1.

u_2 has no shocks, it trivially satisfies the entropy condition.

we now check the entropy condition for u_1:

$c(u_1^+) = 2^2 = 4 > 7/3$

$c(u_1^-) = 1^2 = 1 < 7/3$

u_1 does not satisfy the entropy condition.

u_2 is the physical solution.

Outline

Frame Title

A General linear second-order PDE in two independent variables has the form:

$$L[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

The discriminant:

$$\delta(L)(x_0, y_0) := b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0)$$

↑ local property

$\delta(L)(x_0, y_0) > 0$: hyperbolic — wave equation : $u_{tt} - u_{xx} = 0$

$\delta(L)(x_0, y_0) = 0$: parabolic — heat equation : $u_t - u_{xx} = 0$

$\delta(L)(x_0, y_0) < 0$: elliptic — Laplace equation : $u_{xx} + u_{yy} = 0$

Outline

Tips for Serie 5

1. Weak solutions
   - (a) Remember the chain rule.
   - (b) Check the Rankie-Hugoniot condition.
     What did we do wrong? When can we apply the chain rule?
2. Balance laws
   - (a) Write $u(x, y)$ as some terms plus the integral of g.
   - (b) Check the form of the characteristics.
3. Multiple choice
   - Check the definition in the script.
4. Weak solutions II
   - (b) Check the two inequalities of the entropy condition separately.
5. Finding shock waves
   - Done in the exercise session.

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References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. “Analysis in Mechanical Engineering” by Leon van Dommelen
4. Standford math220a handouts