Analysis 3
Exercise 4

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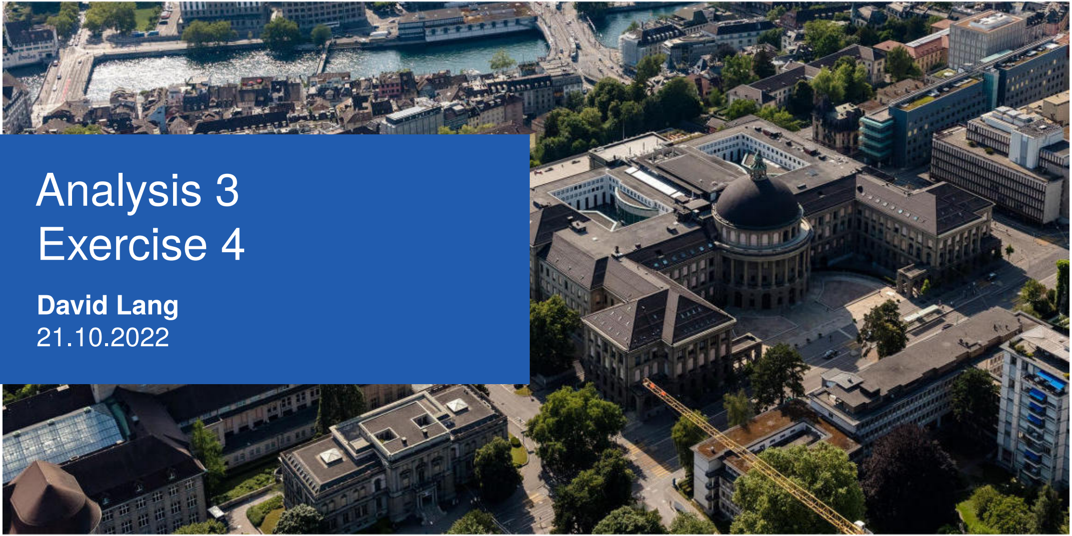

Outline

1. Serie 3 Review
2. Course Overview
3. Conservation Laws
4. Shock Formation and Critical time
5. Weak solution and Rankine-Hugoniot Condition
6. Tips for Serie 4

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Serie 3 Review

1. Characteristic method and initial conditions
   - Always check if the solution fits the initial curve.
2. Method of characteristic, local and global existence
   - Well done
3. Multiple choice
   - $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
   - $\Gamma(s) = (acos(s), bsin(s), u(t=0, s))$
4. Characteristic method and transversality condition
   - Show that w is constant along the characteristic curves.
   - $\frac{\partial}{\partial t} w = 0$

Outline

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - **Conservation laws**
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

Motivation for Conservation laws: Traffic Flow Problem [Extra]

Let $u(x,t)$ be the density of cars at point x, time t.
Therefore, the total number of cars between points x_1 and x_2 at time t can be represented by:

$$\int_{x_1}^{x_2} u(x,t)\,dx$$

The rate of change in the number of cars between points x_1 and x_2 at time t is given by

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

where f represents the flow rate onto and off the street.

$$\int_{x_1}^{x_2} u_t(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

and, therefore

$$\frac{\int_{x_1}^{x_2} u_t(x,t)\,dx}{x_2 - x_1} = \frac{f(u(x_1,t)) - f(u(x_2,t))}{x_2 - x_1}$$

Taking the limit as $x2 \to x1$, we get

$$u_t = -[f(u)]_x$$

General Formulation for Conservation laws

We use x as a spatial variable and y as a temporal variable, so $y > 0$.
We look for $u(x, y) : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ such that

$$u_y + \frac{\partial}{\partial x} f(u) = 0$$

$$u_y + c(u) u_x = 0$$

We often have the initial condition at y (time) = 0, i.e. $u(x, y = 0) = h(x)$.

Examples

Transport equation:
$u_y + c u_x = 0$

Burgers' equation:
$u_y + u u_x = 0$

Derivation of Burgers Equation [Extra]

Consider a one-dimensional medium of particles moving along a line by inertia, so that the velocity of each particle remains constant.

We denote the velocity of the particle at the point x at time t by $u(t, x)$.

We then write Newton's equation: the acceleration of the particle equals zero.

If $x = \varphi(t)$ is the motion of a particle, then

$$\dot{\varphi} = u(t, \varphi(t)), \ddot{\varphi} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\dot{\varphi} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

Thus the velocity field of a medium consisting of non-interacting particles satisfies the quasi-linear equation

$$u_t + uu_x = 0$$

Conservation laws

These types of problems PDEs can be solved with the method of characteristics since they are 1st order Quaslinear PDEs.

Properties of conservation laws:
Solutions of conservation laws are constant along their characteristics, which are straight lines.

For each $s \in \mathbb{R}$ the characteristic through a point $(s, 0)$ is the line in the (x, y) plane going through $(s, 0)$ with slope $1/c(u_0(s))$ and on this line u is equal to the constant $u_0(s)$.

Graphical Illustrations

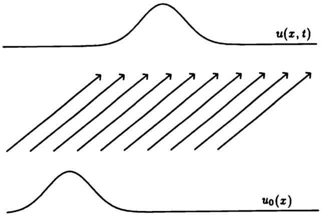

Figure 3.1. Characteristics and solution for the advection equation.

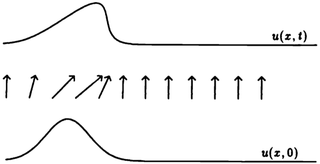

Figure 3.3. Characteristics and solution for Burgers' equation (small t).

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Shock Formation

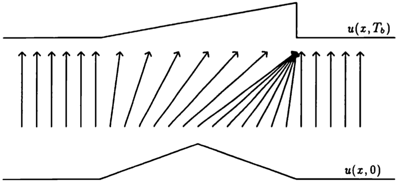


Figure 3.4. Shock formation in Burgers' equation.

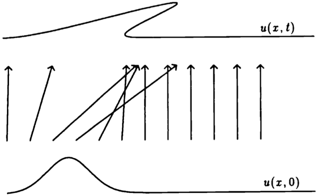


Figure 3.5. Triple-valued solution to Burgers' equation at time $t > T_b$.

Critical Time

If $c(u_0(s))_s < 0$, then there exists a time when the characteristics cross.

Faster characteristics “catch up” slower characteristics.

If $c(u_0(s))$ is never decreasing, there are no singularities.

Example

$$\begin{cases} u_y + uu_x = 0 \\ u(x, y = 0) = arctan(x) \end{cases}$$

$$\begin{cases} u_y + uu_x = 0 \\ u(x, y = 0) = 1 & for \;\; x < -\pi/2 \\ u(x, y = 0) = -sin(x) & for \; |x| \leq \pi/2 \\ u(x, y = 0) = -1 & for \;\; x > \pi/2 \end{cases}$$

Outline

Weak solution and Rankine-Hugoniot Condition

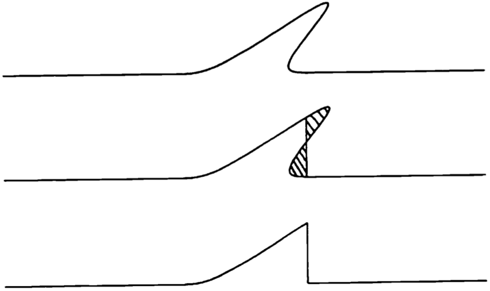
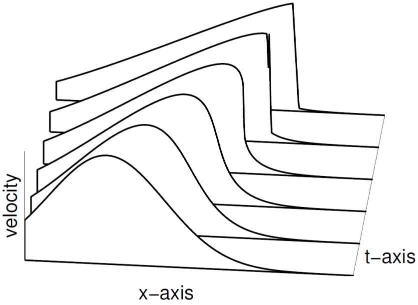
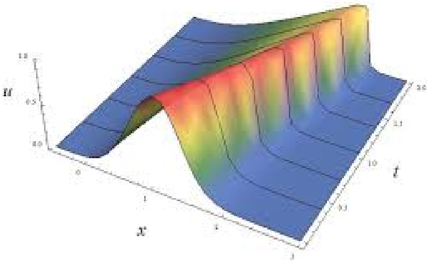


Figure 3.13. Equal area rule for shock location.

$$\int_a^b u(x, y_2)\, dx - \int_a^b u(x, y_1)\, dx$$
$$= -\int_a^b [f(u(b, y)) - f(u(a, y))]\, dy$$

Illustrations





Example

$$\begin{cases} u_y + u^2 u_x = 0 \\ u(x, y=0) = 3 \; for \; x < 0 \\ u(x, y=0) = 1 \; for \; x > 0 \end{cases}$$

Outline

Tips for Serie 4

1. Conservation laws and critical times
   - (a) See Exercise example
   - (b) Change of Variables: $s^2 = \tau$
   - (c) Shock or no shock?
2. Multiple Choice
   - (c) Critical time and characteristics
3. Weak solutions
   - Check the definition in the script.

Additional Reading:

http://www.clawpack.org/riemann_book/html/Burgers.html

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. “Burgers equation” by Mikel Landajuela
4. “Numerical Methods for Conservation Laws, Lectures in Mathematics, ETH-Zurich” by Randall J. LeVeque
5. “Ordinary Differential Equations” by Vladimir I. Arnol’d
6. “The shocking behaviour of moving fluids” by Antoine Nectoux.
7. Standford math220a handouts