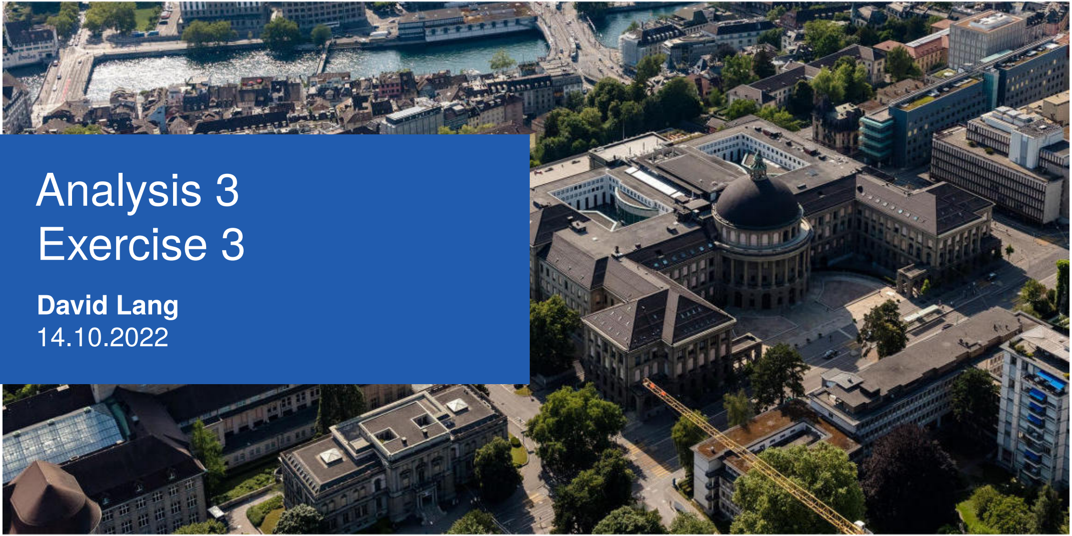

Analysis 3
Exercise 3

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Outline

1. Serie 2 Review
2. Course Overview
3. Implicit function theorem
4. The existence and uniqueness theorem
5. Global Existence
6. Tips for Serie 3

Outline

General procedure for Method of characteristics

1. Find a, b, c
2. Find $\Gamma(s) = \begin{bmatrix} x(0,s) \\ y(0,s) \\ \tilde{u}(0,s) \end{bmatrix}$

   i.e. $u(x,0) = f(x) \; then \; \Gamma(s) = \begin{bmatrix} s \\ 0 \\ f(s) \end{bmatrix}$

3. $\begin{cases} \frac{d}{dt}x = a, \; x_0 = x(0,s) \\ \frac{d}{dt}y = b, \; y_0 = y(0,s) \\ \frac{d}{dt}\tilde{u} = c, \; \tilde{u_0} = u(0,s) \end{cases}$
4. Solve the equations
5. Plug $s(x,y), t(x,y)$ into $\tilde{u}(s,t)$.

Serie 2 Review

1. Method of characteristics I
   - (b) $\begin{cases} xu_x + (x+y)u_y = 1 \\ u(1,y) = y^2 \end{cases}$
2. Method of characteristics II
   - We will see today the condition when a map is invertible.
3. Multiple choice
   - (d) $x^3 + 1 = y$
4. Find a solution
   - $sin(s)^2 + cos(s)^2 = 1$

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - **Method of characteristics**
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

Implicit function theorem with 2 variables

Let $F(x,y) \in C^1$ in a neighborhood of (x_0, y_0) such that

1. $F(x_0, y_0) = 0$
2. $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$

Then there is an implicit but unique function $y = f(x) \in C^1$ in a neighbourhood of x_0 such that

1. $f(x_0) = y_0$
2. $F(x, f(x)) = 0$ for every x near x_0
3. $f'(x) = -\frac{\frac{\partial F}{\partial x}(x,f(x))}{\frac{\partial F}{\partial y}(x,f(x))}$ in the neigborhood

Implicit function theorem for n+1 variables

Denote $\vec{x} = (x_1, x_2, ..., x_n)$
Let $F(\vec{x}, y) \in C^1$ in a neighborhood of $(\vec{x_0}, y_0)$ such that

1. $F(\vec{x_0}, y_0) = 0$
2. $\frac{\partial F}{\partial y}(\vec{x_0}, y_0) \neq 0$

Then there is an implicit but unique function $y = f(\vec{x}) \in C^1$ in a neighbourhood of $\vec{x_0}$ such that

1. $f(\vec{x_0}) = y_0$
2. $F(\vec{x}, f(\vec{x})) = 0$ for every $\vec{x}$ near $\vec{x_0}$
3. $\frac{\partial f}{\partial x_i} = -\frac{\frac{\partial F}{\partial x_i}(\vec{x}, f(\vec{x}))}{\frac{\partial F}{\partial y}(\vec{x}, f(\vec{x}))}$ in the neigborhood

Implicit function theorem for systems of variables

$$\begin{cases} F_1(x, y, s, t) = 0 \\ F_2(x, y, s, t) = 0 \end{cases}$$

$$det\begin{pmatrix} \frac{\partial F_1}{\partial t} & \frac{\partial F_1}{\partial s} \\ \frac{\partial F_2}{\partial t} & \frac{\partial F_3}{\partial s} \end{pmatrix} \neq 0$$

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The existence and uniqueness theorem

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ \Gamma(s) = (x_0(s), y_0(s), u_0(s)) \end{cases}$$

The transversality condition holds, i.e.

$$det \begin{bmatrix} \frac{\partial x}{\partial t}(0,s_0) & \frac{\partial y}{\partial t}(0,s_0) \\ \frac{\partial x}{\partial s}(0,s_0) & \frac{\partial y}{\partial s}(0,s_0) \end{bmatrix} \neq 0$$

Then there exists a unique solution u in a neighborhood of $(x_0(s_0), y_0(s_0))$.

Example

$$\begin{cases} 3uu_x + u_y = u \\ u(x,0) = x \end{cases}$$

What if the transversality condition does not hold?

Outline

Global Existence

1. The characteristics intersect Γ more than once.

2. The characteristics intersect with themselves.

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Tips for Serie 3

1. Characteristic method and initial conditions
   - (a) 2.1 (d)
   - (d) case studies on $x < 0, x > 0$.
   - (c) $f(s)$ is defined only when $s > 0$.
2. Method of characteristic, local and global existence
3. Multiple Choice
   - (c) When is $sin(s)$ maximal and minimal?
4. Characteristic method and transversality condition
   - (b) What is the relationship between (s, s, s) and (se^t, se^t, se^t) ?

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Additional reading: Textbook

- 2.4 Examples of the characteristics method
- 2.5 The existence and uniqueness theorem

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. Analysis Skript by Prof. Struwe