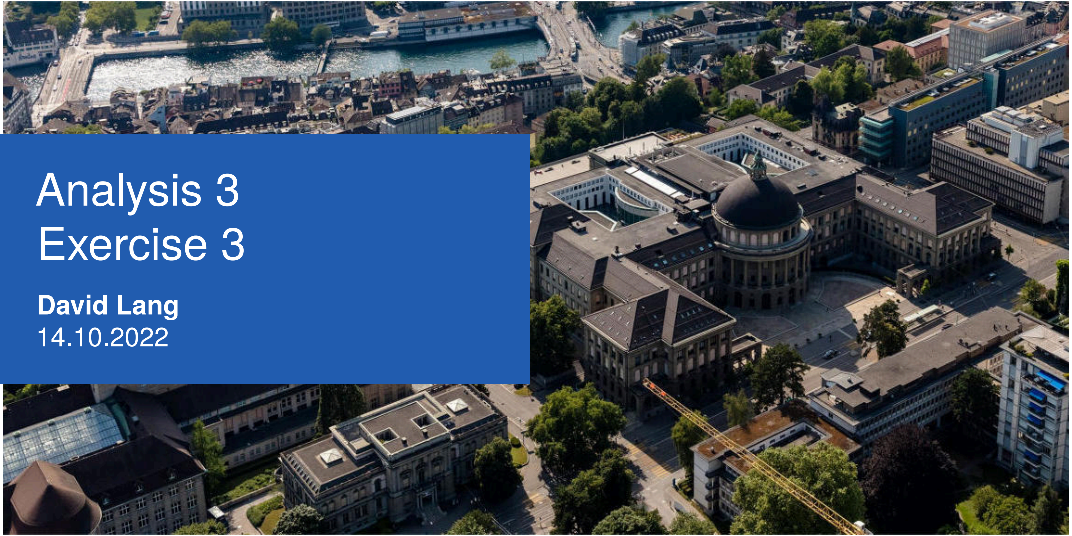

Analysis 3
Exercise 3

David Lang

14.10.2022

Outline

Outline

1. Serie 2 Review
2. Course Overview
3. Implicit function theorem
4. The existence and uniqueness theorem
5. Global Existence
6. Tips for Serie 3

General procedure for Method of characteristics

1. Find a, b, c
2. Find $\Gamma(s) = \begin{bmatrix} x(0,s) \\ y(0,s) \\ \tilde{u}(0,s) \end{bmatrix}$

   i.e. $u(x,0) = f(x)\ then\ \Gamma(s) = \begin{bmatrix} s \\ 0 \\ f(s) \end{bmatrix}$
3. $\begin{cases} \frac{d}{dt}x = a,\ x_0 = x(0,s) \\ \frac{d}{dt}y = b,\ y_0 = y(0,s) \\ \frac{d}{dt}\tilde{u} = c,\ \tilde{u_0} = u(0,s) \end{cases}$
4. Solve the equations
5. Plug $s(x,y), t(x,y)$ into $\tilde{u}(s,t)$.

Serie 2 Review

1. Method of characteristics I
   - (b) $\begin{cases} xu_x + (x+y)u_y = 1 \\ u(1,y) = y^2 \end{cases}$
2. Method of characteristics II
   - We will see today the condition when a map is invertible.
3. Multiple choice
   - (d) $x^3 + 1 = y$, $u=2$
4. Find a solution
   - $sin(s)^2 + cos(s)^2 = 1$

$\begin{cases} a = x \\ b = x+y \\ c = 1 \end{cases}$ $\quad T(s) = \begin{pmatrix} 1 \\ s \\ s^2 \end{pmatrix}$ $\quad \begin{cases} x_t = x \\ y_t = x+y \\ \hat{u}_t = 1 \end{cases}$ $\quad x(t=0,s)=1$, $y(t=0,s)=s$, $\hat{u}(t=0,s)=s^2$

$x = e^t$ $\quad y_t = x+y$ $\quad \hat{u} = t + s^2$

$y_t - y = e^t$

$(e^{-t}y)' = 1$

$y = (s+t)e^t$

$\begin{cases} x = e^t \\ y = (s+t)e^t \end{cases} \Rightarrow \begin{cases} t = \ln(x) \\ s = \frac{y}{x} - \ln(x) \end{cases}$

$u(x,y) = \ln(x) + \left(\frac{y}{x} - \ln(x)\right)^2$

$T(s) = \begin{pmatrix} s \\ s^3+1 \\ 2 \end{pmatrix}$ $\quad T(s) = \begin{pmatrix} s^{1/3} \\ s+1 \\ 2 \end{pmatrix}$ $\quad (s^{1/3})^3 + 1 = s+1$ ✓

not unique!

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - **Method of characteristics**
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

Implicit function theorem with 2 variables

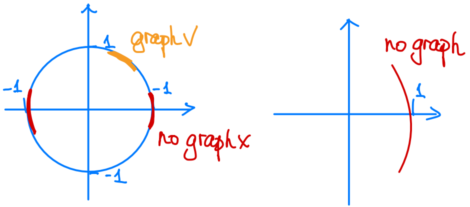
Let $F(x,y) \in C^1$ in a neighborhood of (x_0, y_0) such that

1. $F(x_0, y_0) = 0$
2. $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$

Then there is an implicit but unique function $y = f(x) \in C^1$ in a neighbourhood of x_0 such that

1. $f(x_0) = y_0$
2. $F(x, f(x)) = 0$ for every x near x_0
3. $f'(x) = -\frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}$ in the neigborhood

relationship: $x^2 + y^2 = 1$ ← not a function



given: $F(x,y) = x^2 + y^2 - 1$ goal: $y = f(x)$

$\frac{\partial F}{\partial y} = 2y \neq 0$ $y \neq 0$ $(x,y) \neq (\pm 1, 0)$

Implicit function theorem for n+1 variables

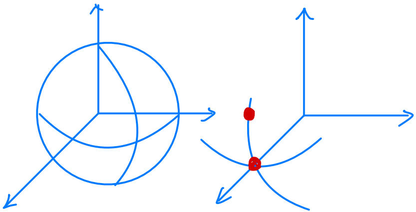
relationship: $x^2+y^2+z^2=1$ ← no function

Denote $\vec{x} = (x_1, x_2, ..., x_n)$
Let $F(\vec{x}, y) \in C^1$ in a neighborhood of $(\vec{x_0}, y_0)$ such that

1. $F(\vec{x_0}, y_0) = 0$
2. $\frac{\partial F}{\partial y}(\vec{x_0}, y_0) \neq 0$

Then there is an implicit but unique function $y = f(\vec{x}) \in C^1$ in a neighbourhood of $\vec{x_0}$ such that

1. $f(\vec{x_0}) = y_0$
2. $F(\vec{x}, f(\vec{x})) = 0$ for every $\vec{x}$ near $\vec{x_0}$
3. $\frac{\partial f}{\partial x_i} = -\frac{\frac{\partial F}{\partial x_i}(\vec{x}, f(\vec{x}))}{\frac{\partial F}{\partial y}(\vec{x}, f(\vec{x}))}$ in the neigborhood



given: $F(x,y,z) = x^2+y^2+z^2-1$

goal: $y = f(x,z)$

$\frac{\partial F}{\partial y} = 2y \neq 0$ $\quad$ $y \neq 0$

Implicit function theorem for systems of variables

$$\begin{cases} F_1(x,y,s,t) = 0 \\ F_2(x,y,s,t) = 0 \end{cases}$$

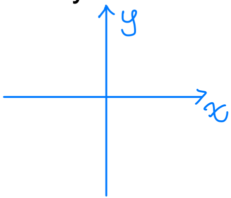
$$det\begin{pmatrix} \frac{\partial F_1}{\partial t} & \frac{\partial F_1}{\partial s} \\ \frac{\partial F_2}{\partial t} & \frac{\partial F_3}{\partial s} \end{pmatrix} \neq 0$$

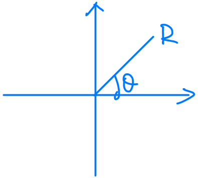
Our case:

given $\begin{cases} x(s,t) = \dots \\ y(s,t) = \dots \end{cases}$

$\Downarrow$?

find $\begin{cases} s(x,y) = \dots \\ t(x,y) = \dots \end{cases}$





given $\begin{cases} x(R,\theta) = R\cos(\theta) \\ y(R,\theta) = R\sin(\theta) \end{cases}$ find $\begin{cases} R(x,y) = \dots \\ \theta(x,y) = \dots \end{cases}$

$$|J| = \left\| \begin{bmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \cos(\theta) & -R\sin(\theta) \\ \sin(\theta) & R\cos(\theta) \end{bmatrix} \right\| = R^2(\cos^2(\theta) + \sin^2(\theta)) = R^2$$

The conversion is possible if $R \neq 0$

$R = 0$ (origin): θ is not well-defined

Outline

extra:

$\det(A) = \det(A^T)$

proof:

$A = QR$

$\det(A^T) = \det(R^T Q^T) \overset{①}{=} \det(R)\det(Q^T)$

$\overset{②}{=} \det(R)\det(Q^{-1}) \overset{③}{=} \frac{\det(R)}{\det(Q)} \overset{④}{=} \det(Q)\det(R)$

$= \det(QR) = \det(A)$

①: determinant of a triangular matrix is the product of the diagonal elements which remain unchanged

②: The transpose of an orthogonal matrix is its inverse

③: $\det(B^{-1}) = \frac{1}{\det(B)}$

④: the determinant of an orthogonal matrix is either -1 or 1.

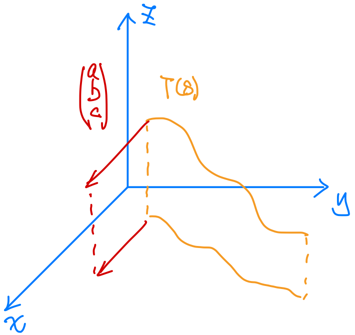
The existence and uniqueness theorem ← local

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ \Gamma(s) = (x_0(s), y_0(s), u_0(s)) \end{cases}$$

The transversality condition holds, i.e.

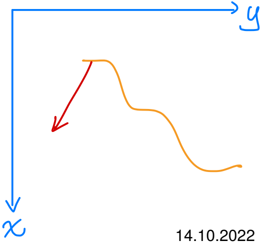
$$det \begin{bmatrix} \frac{\partial x}{\partial t}(0,s_0) & \frac{\partial y}{\partial t}(0,s_0) \\ \frac{\partial x}{\partial s}(0,s_0) & \frac{\partial y}{\partial s}(0,s_0) \end{bmatrix} \neq 0$$

Then there exists a unique solution u in a neighborhood of $(x_0(s_0), y_0(s_0))$.



$$\det\begin{pmatrix} a & b \\ x_0(s)_s & y_0(s)_s \end{pmatrix} = \det\begin{pmatrix} a(0,s) & b(0,s) \\ \frac{d}{ds}x(t=0,s) & \frac{d}{ds}y(t=0,s) \end{pmatrix} \neq 0$$

$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \frac{d}{ds}x \\ \frac{d}{ds}y \end{pmatrix}$ not linearly dependent, not tangental



Example

$$\begin{cases} 3uu_x + u_y = u \\ u(x,0) = x \end{cases}$$

$$\begin{cases} a = 3\tilde{u} \\ b = 1 \\ c = \tilde{u} \end{cases} \qquad T(s) = \begin{pmatrix} x(t=0,s) \\ y(t=0,s) \\ \tilde{u}(t=0,s) \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix}$$

$$\det \begin{vmatrix} a(t=0,s) & b(t=0,s) \\ \frac{d}{ds}x(t=0,s) & \frac{d}{ds}y(t=0,s) \end{vmatrix} = \det \begin{vmatrix} 3\tilde{u}(t=0,s) & 1 \\ 1 & 0 \end{vmatrix} = \det \begin{vmatrix} 3s & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

transversality condition is satisfied → exist a locally unique solution

$$\begin{cases} x_t = \tilde{u}(t,s) & x(0,s) = s \\ y_t = 1 & y(0,s) = 0 \\ \tilde{u}_t = u & \tilde{u}(0,s) = s \end{cases}$$

since the equation for x depends on $\tilde{u}$, we shall first solve $\tilde{u}$. $\tilde{u}(t,s) = se^t$

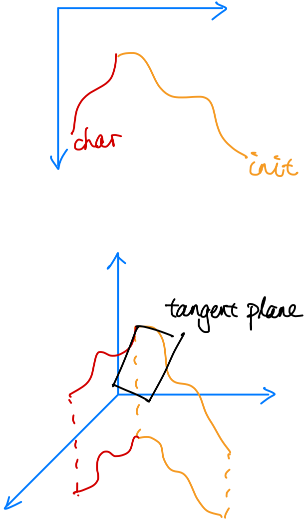
$x_t = se^t$, $x(0,s) = s \Rightarrow x(t,s) = 3se^t - 2s$

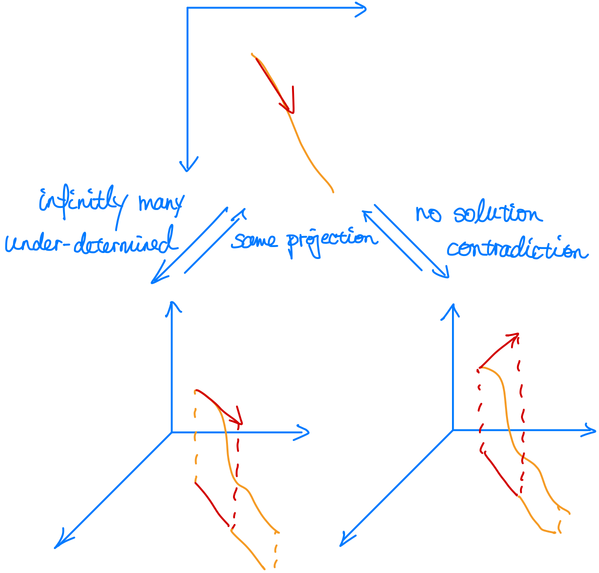
$y = t$

$$\begin{cases} x(t,s) = 3se^t - 2s \\ y(t,s) = t \end{cases} \Rightarrow \begin{cases} s = \dfrac{x}{3e^y - 2} \\ t = y \end{cases}$$

$$u(x,y) = s(x,y)e^{t(x,y)} = \frac{x}{3e^y - 2} e^y$$

What if the transversality condition does not hold?

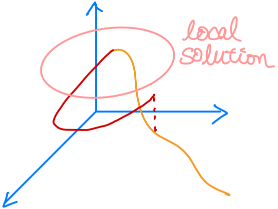




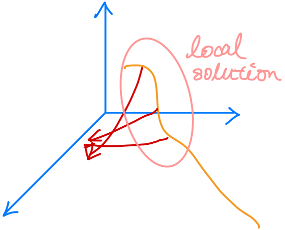
Outline

Global Existence depends on the domain

1. The characteristics intersect Γ more than once.



2. The characteristics intersect with themselves.



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Tips for Serie 3

1. Characteristic method and initial conditions
   - (a) 2.1 (d)
   - (d) case studies on $x < 0, x > 0$.
   - (c) $f(s)$ is defined only when $s > 0$.
2. Method of characteristic, local and global existence
3. Multiple Choice
   - (c) When is $sin(s)$ maximal and minimal?
4. Characteristic method and transversality condition
   - (b) What is the relationship between (s, s, s) and (se^t, se^t, se^t) ?

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Additional reading: Textbook

- 2.4 Examples of the characteristics method
- 2.5 The existence and uniqueness theorem

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. Analysis Skript by Prof. Struwe