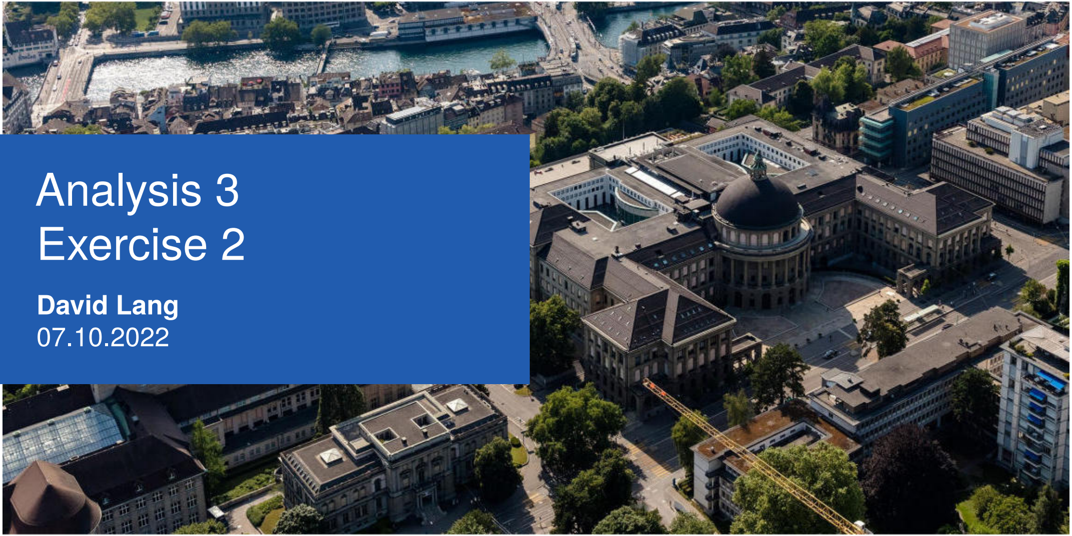


Analysis 3
Exercise 2

David Lang

07.10.2022

Outline

1. Serie 1 Review
2. Course Overview
3. Graph, tangent and normal
4. Method of characteristics
5. Examples
6. Tips for Exercise 2

Outline

Gradient, Divergence, Curl and Laplacian

Gradient
(scalar to vector)

Divergence
(vector to scalar)

Curl
(vector to vector)

Laplacian
(scalar to scalar)

Serie 1 Review

1. Classification of PDEs
   - $(c)\ (u_{xx}+1)^3 = x^3 + 2$
2. Solutions to ODEs
3. Nonexistence of solutions
4. Existence of infinite solutions
5. Multiple Choice
   - $(c)\ v := e^u\ \&\ \Delta u + \nabla u \cdot \nabla u = 0$
   - $(d)\ div(\nabla(u^2)) = u$
6. Classification of PDEs
   - $(c)\ e^{\Delta u} = u$
7. Solutions to PDEs

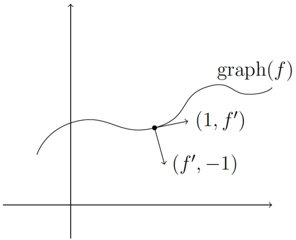
Outline

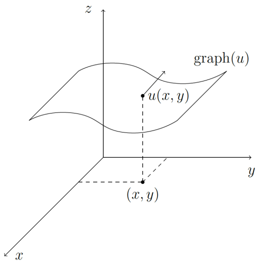
Course Overview

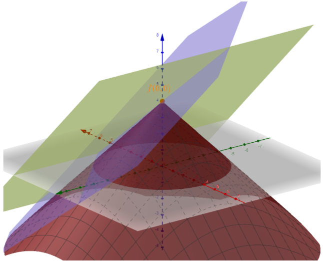
- 1st order PDEs
  - Quasilinear first order PDEs
    - **Method of characteristics**
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

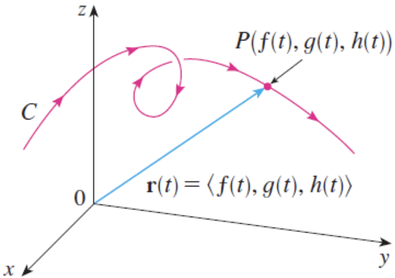
Graph, tangent and normal





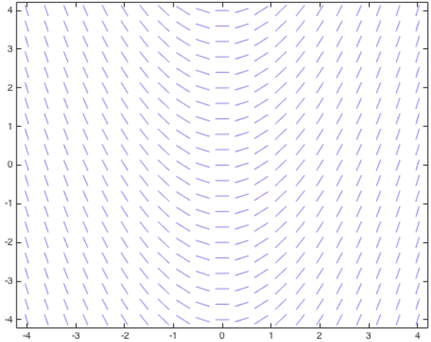


Parameterize a curve in 3d



Outline

From Vector Field to ODE



First-order Eqautions

We focus on two-dimensional real-valued function: $u(x, y)$.

General form for the PDE: $F(x, y, u, u_x, u_y) = 0$.

$graph(u) := (x, y, u(x, y))\ in\ \mathbb{R}^3$

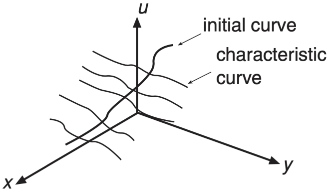
Quasilinear: $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$

Method of characteristics

Results from the last slide:

$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the tangent plane of $u(x, y)$ and is related to the derivative of the $u(x, y)$.

Goal: find the points $(x; y; u) \in S$, such that $(a(x; y); b(x; y); c(x; y; u)) \in T_{(x;y;u)}S$.



General procedure for Method of characteristics

1. Find a, b, c
2. Find $\Gamma(s) = \begin{bmatrix} x(0,s) \\ y(0,s) \\ \tilde{u}(0,s) \end{bmatrix}$

   i.e. $u(x,0) = f(x)\ then\ \Gamma(s) = \begin{bmatrix} s \\ 0 \\ f(s) \end{bmatrix}$
3. $\begin{cases} \frac{d}{dt}x = a,\ x_0 = x(0,s) \\ \frac{d}{dt}y = b,\ y_0 = y(0,s) \\ \frac{d}{dt}\tilde{u} = c,\ \tilde{u_0} = u(0,s) \end{cases}$
4. Solve the equations
5. Plug $s(x,y), t(x,y)$ into $\tilde{u}(s,t)$.

Outline

Example 1

$$\begin{cases} x^2 u_x - (y^2+1) u_y = x^3 u \\ u(1,y) = 2 \end{cases}$$

Example 2

$$\begin{cases} u_x + xyu_y = xu \\ u(0,y) = f(y) \end{cases}$$

Outline

1. Serie 1 Review
2. Course Overview
3. Graph, tangent and normal
4. Method of characteristics
5. Examples
6. Tips for Exercise 2

Tips for Exercise 2

1. Method of characteristics I
   - (c) for y use separation of variables
   - (d) $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$
     Differentiate again to uncouple the equations.
2. Method of characteristics II
   - (b): Is there a way to express t and s in terms of x and y?
     Why is this the case?
     Which of the 3 different cases is it according to the lecture note?
3. Multiple Choice
   - $(c)\ (\frac{f}{g})' = (fg^{-1})' = \frac{f'}{g} - \frac{fg'}{g^2}$
4. Find a solution
   - What is $x^2 + y^2$?

ETH zürich

Peers found useful:

1. Method of Characteristics:
   Chapters 2.3 and 2.4 of the textbook.

Additional reading:

1. Extra02 on the course website: An example of solution.

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. University of Washington Lecture notes