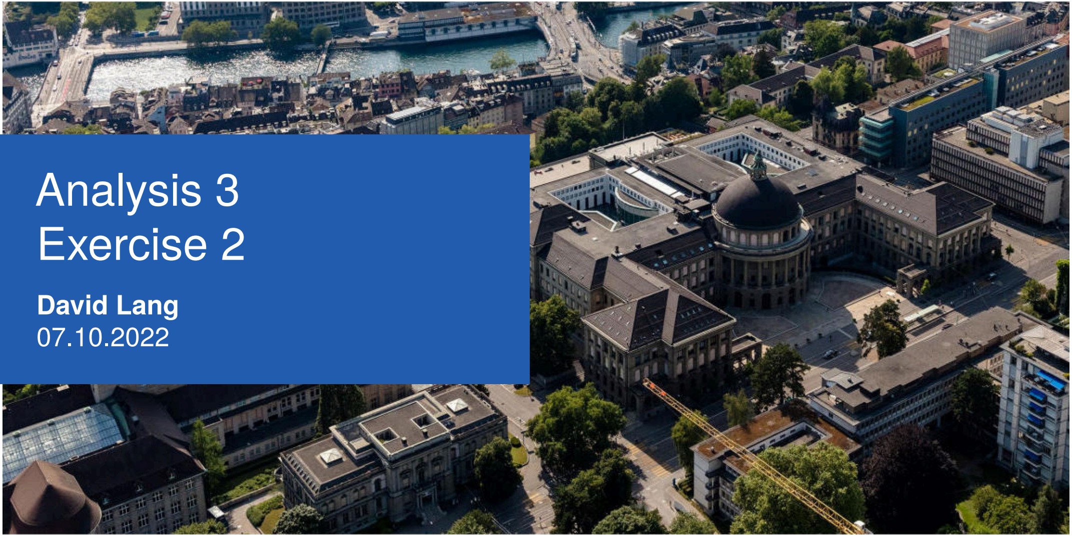


Analysis 3
Exercise 2

David Lang

07.10.2022

Outline

1. Serie 1 Review
2. Course Overview
3. Graph, tangent and normal
4. Method of characteristics
5. Examples
6. Tips for Exercise 2

Outline

Gradient, Divergence, Curl and Laplacian

Gradient
(scalar to vector)

$u(x,y,z)$

$$\mathrm{Grad}(u) = \boxed{\nabla u} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix}$$

Divergence
(vector to scalar)

$$\begin{pmatrix} u_1(x,y,z) \\ u_2(x,y,z) \\ u_3(x,y,z) \end{pmatrix}$$

$$\mathrm{Div}(\vec{u}) = \nabla \cdot \vec{u} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$$

Curl
(vector to vector)

$$\begin{pmatrix} u_1(x,y,z) \\ u_2(x,y,z) \\ u_3(x,y,z) \end{pmatrix}$$

$$\mathrm{Curl}(\vec{u}) = \nabla \times \vec{u} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \\ \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \\ \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \end{pmatrix}$$

Laplacian
(scalar to scalar)

$u(x,y,z)$

$$\Delta u = \nabla^2 u = \nabla \cdot (\boxed{\nabla u}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Serie 1 Review

1. Classification of PDEs
   - $(c)\ (u_{xx}+1)^3 = x^3+2$
2. Solutions to ODEs
3. Nonexistence of solutions
4. Existence of infinite solutions
5. Multiple Choice
   - $(c)\ v := e^u\ \&\ \Delta u + \nabla u \cdot \nabla u = 0$
   - $(d)\ div(\nabla(u^2)) = u$
6. Classification of PDEs
   - $(c)\ e^{\Delta u} = u$
7. Solutions to PDEs

$u_{xx}+1 = \sqrt[3]{x^3+2}$ linear (transform)

$u = \ln(v)$

$\Delta(\ln(v)) + \nabla(\ln(v)) \cdot \nabla(\ln(v))$

$= (\ln(v))_{xx} + (\ln(v))_{yy} + (\ln(v))_x^2 + (\ln(v))_y^2$

$= \left(\frac{v_x}{v}\right)_x + \left(\frac{v_y}{v}\right)_y + \left(\frac{v_x}{v}\right)^2 + \left(\frac{v_y}{v}\right)^2$

$= \dots$

chain rule: $\nabla(u^2) = \nabla u \cdot 2u$

$\Delta u = \ln(u)$ $\quad u_{xx}+u_{yy} = \ln(u)$ Quasilinear

Outline

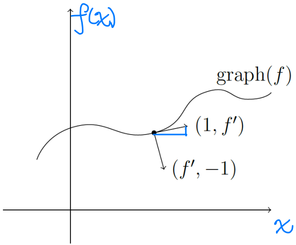
1. Serie 1 Review
2. Course Overview
3. Graph, tangent and normal
4. Method of characteristics
5. Examples
6. Tips for Exercise 2

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - **Method of characteristics**
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

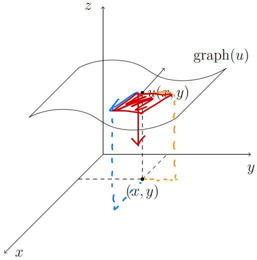
Outline

Graph, tangent and normal



tangent

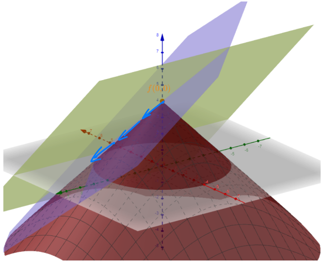
$$\begin{pmatrix}1\\f'\end{pmatrix} \perp \begin{pmatrix}f'\\-1\end{pmatrix}$$



$$\begin{pmatrix}0\\1\\u_y\end{pmatrix} \times \begin{pmatrix}1\\0\\u_x\end{pmatrix} = \begin{pmatrix}u_x\\u_y\\-1\end{pmatrix}$$

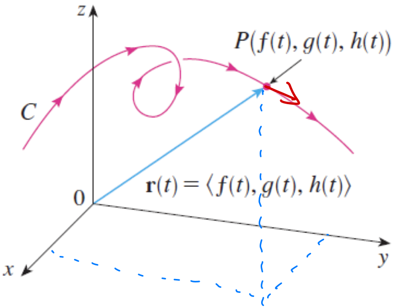
↓

downward pointing normal vector



by "following" the tangent plane, we "move" on the solution surface.

Parameterize a curve in 3d



at each time t, we need the x, y, z positions

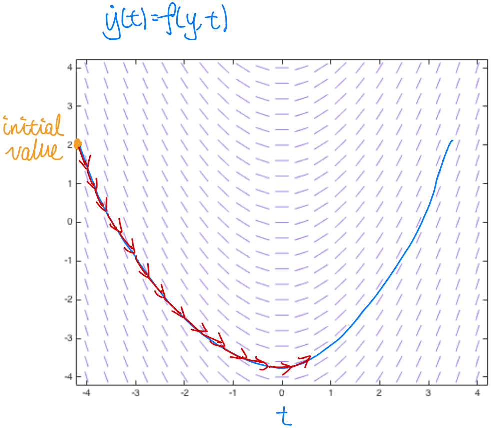
if we know $u(x, y)$

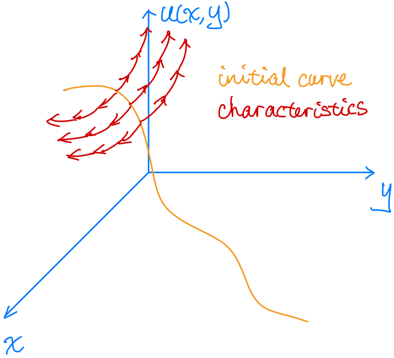
$$\vec{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ u(f(t), g(t)) \end{pmatrix}$$

$\frac{d\vec{r}(t)}{dt}$: velocity / tangent

Outline

From Vector Field to ODE





First-order Eqations

We focus on two-dimensional real-valued function: $u(x, y)$.

General form for the PDE: $F(x, y, u, u_x, u_y) = 0$.

$graph(u) := (x, y, u(x, y))\ in\ \mathbb{R}^3$

Quasilinear: $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \perp \begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} u_x \\ u_y \\ -1 \end{bmatrix} \perp \text{tangent plane}$$

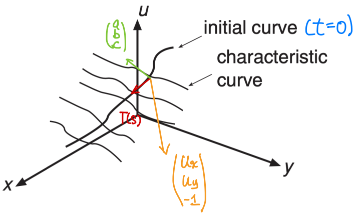
$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ is in the tangent plane}$$

Method of characteristics

Results from the last slide:

$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the tangent plane of $u(x, y)$ and is related to the derivative of the $u(x, y)$.

Goal: find the points $(x; y; u) \in S$, such that $(a(x; y); b(x; y); c(x; y; u)) \in T_{(x;y;u)}S$.



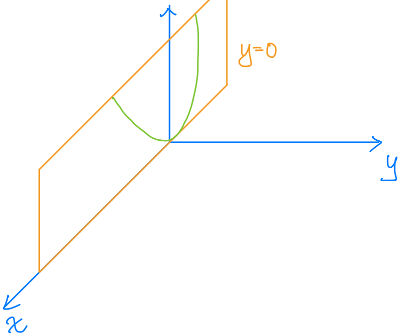
initial curve (t=0) parameterized by s.

along the initial curve t=0, s changes

along the characteristics, t changes

$$\begin{cases} \frac{dx}{dt} = a \\ \frac{dy}{dt} = b \\ \frac{du}{dt} = c \end{cases}$$

General procedure for Method of characteristics



1. Find a, b, c
2. Find $\Gamma(s) = \begin{bmatrix} x(0,s) \\ y(0,s) \\ \tilde{u}(0,s) \end{bmatrix}$

   i.e. $u(x,0) = f(x)\ then\ \Gamma(s) = \begin{bmatrix} s \\ 0 \\ f(s) \end{bmatrix}$
3. $\begin{cases} \frac{d}{dt}x = a, \ x_0 = x(0,s) \\ \frac{d}{dt}y = b, \ y_0 = y(0,s) \\ \frac{d}{dt}\tilde{u} = c, \ \tilde{u_0} = u(0,s) \end{cases}$
4. Solve the equations
5. Plug $s(x,y), t(x,y)$ into $\tilde{u}(s,t)$. $\rightarrow u(x,y)$

Outline

1. Serie 1 Review
2. Course Overview
3. Graph, tangent and normal
4. Method of characteristics
5. Examples
6. Tips for Exercise 2

Example 1

$$\begin{cases} x^2 u_x - (y^2+1)u_y = x^3 u \\ u(1,y) = 2 \end{cases}$$

$$\begin{cases} a = x^2 \\ b = -(y^2+1) \\ c = x^3 \hat{u} \end{cases} \qquad \Gamma(s) = \begin{pmatrix} 1 \\ s \\ 2 \end{pmatrix} \qquad \begin{cases} \frac{dx}{dt} = x^2 & x(t=0,s)=1 \\ \frac{dy}{dt} = -y^2-1 & y(t=0,s)=s \\ \frac{d\hat{u}}{dt} = \hat{u}x^3 & \hat{u}(t=0,s)=2 \end{cases}$$

For x:

$$\frac{1}{x^2}dx = 1dt$$

$$\int \frac{1}{x^2}dx = \int 1dt$$

$$-\frac{1}{x} = t + C_1(s)$$

$$x = -\frac{1}{t+C_1(s)}$$

$$x(t=0,s) = 1$$

$$-\frac{1}{0+C_1(s)} = 1 \quad C_1(s) = -1$$

$$x(t,s) = \frac{1}{1-t}$$

For y:

$$\frac{1}{y^2+1}dy = -1dt$$

$$\int \frac{1}{y^2+1}dy = \int -1dt$$

$$\arctan(y) = -t + C_2(s)$$

$$y = \tan(-t + C_2(s))$$

$$y(t=0,s) = s$$

$$y(t,s) = \tan(\arctan(s) - t)$$

For $\hat{u}$:

$$\frac{d\hat{u}}{dt} = \hat{u}x^3 = \hat{u}\left(\frac{1}{1-t}\right)^3$$

$$\frac{1}{\hat{u}}d\hat{u} = \left(\frac{1}{1-t}\right)^3 dt$$

$$\int \frac{1}{\hat{u}}d\hat{u} = \int \left(\frac{1}{1-t}\right)^3 dt$$

$$\ln(\hat{u}) = \frac{1}{2(1-t)^2} + C_3(s)$$

$$\hat{u} = C_4(s)\exp\left(\frac{1}{2(1-t)^2}\right)$$

$$\hat{u}(t=0,s) = 2 \quad C_4(s) = 2e^{-\frac{1}{2}}$$

$$\hat{u}(t,s) = 2e^{-\frac{1}{2}}\exp\left(\frac{1}{2(1-t)^2}\right)$$

$$u(x,y) = \frac{2}{\sqrt{e}}\exp\left(\frac{x^2}{2}\right)$$

(independent of y)

Check:

$$x^2 u_x = x^2 \cdot \frac{2}{\sqrt{e}}\exp\left(\frac{x^2}{2}\right)\cdot x$$

$$= \frac{2}{\sqrt{e}}x^3\exp\left(\frac{x^2}{2}\right)$$

$$-(y^2+1)u_y = 0$$

$$x^3 u = \frac{2}{\sqrt{e}}x^3\exp\left(\frac{x^2}{2}\right)$$

$$\begin{cases} x^2u_x - (y^2+1)u_y = x^3u \ \checkmark \\ u(1,y) = \frac{2}{\sqrt{e}}\exp\left(\frac{1}{2}\right) = 2 \ \checkmark \end{cases}$$

Example 2

$$\begin{cases} u_x + xyu_y = xu \\ u(0,y) = f(y) \end{cases}$$

$$\begin{cases} a = 1 \\ b = xy \\ c = xu \end{cases} \qquad \Gamma(s) = \begin{pmatrix} 0 \\ s \\ f(s) \end{pmatrix}$$

$$\begin{cases} \frac{dx}{dt} = 1 & x(t=0,s) = 0 \\ \frac{dy}{dt} = xy & y(t=0,s) = s \\ \frac{d\hat{u}}{dt} = x\hat{u} & \hat{u}(t=0,s) = f(s) \end{cases}$$

For x:

$x_t = 1$

$x = t + c_1(s)$

$x(t=0,s) = 0 \quad c_1(s) = 0$

$\boxed{x = t}$

For y:

$y_t = xy = \boxed{t}y$

$\boxed{\frac{dy}{dt} = ty}$

$\frac{1}{y}dy = t\,dt$

$\int \frac{1}{y}dy = \int t\,dt$

$\ln(y) = \frac{1}{2}t^2 + c_2(s)$

$y = c_3(s)\exp(\frac{1}{2}t^2)$

$y(t=0,s) = s$

$y = s\exp(\frac{1}{2}t^2)$

For $\hat{u}$:

$\boxed{u_t = x\hat{u} = t\hat{u}}$

Same as y

$\hat{u}(t,s) = f(s)\exp(\frac{1}{2}t^2)$

$\begin{cases} x = t \\ y = s\exp(\frac{1}{2}t^2) \end{cases}$ invert

$\begin{cases} t = x \\ s = y\exp(-\frac{1}{2}x^2) \end{cases}$

$u(x,y) = f(ye^{-\frac{1}{2}x^2})e^{\frac{1}{2}x^2}$

For example:

$u(0,y) = f(y) = y^2$

$u(x,y) = (ye^{-\frac{1}{2}x^2})^2 e^{\frac{1}{2}x^2}$

$= y^2 e^{-x^2} e^{\frac{1}{2}x^2} = y^2 e^{-\frac{1}{2}x^2}$

check:

$u_x = y^2 e^{-\frac{1}{2}x^2} \cdot (-x) = -xy^2 e^{-\frac{1}{2}x^2}$

$xyu_y = xy \cdot 2y \cdot e^{-\frac{1}{2}x^2}$

$= 2xy^2 e^{-\frac{1}{2}x^2}$

$\begin{cases} u_x + xyu_y = xu & \checkmark \\ u(0,y) = y^2 e^0 = y^2 & \checkmark \end{cases}$

Outline

Tips for Exercise 2

1. Method of characteristics I
   - (c) for y use separation of variables
   - (d) $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$
     Differentiate again to uncouple the equations.
2. Method of characteristics II
   - (b): Is there a way to express t and s in terms of x and y?
     Why is this the case?
     Which of the 3 different cases is it according to the lecture note?
3. Multiple Choice
   - $(c)\ (\frac{f}{g})' = (fg^{-1})' = \frac{f'}{g} - \frac{fg'}{g^2}$
4. Find a solution
   - What is $x^2 + y^2$?

Peers found useful:

1. Method of Characteristics:
   Chapters 2.3 and 2.4 of the textbook.

Additional reading:

1. Extra02 on the course website: An example of solution.

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. University of Washington Lecture notes