Analysis 3
Exercise 1

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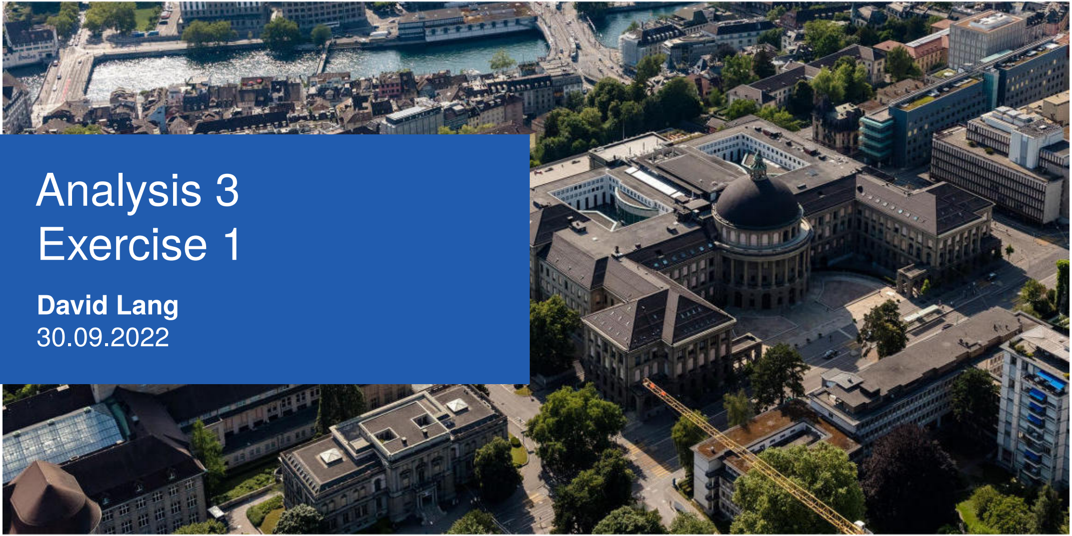


Outline

1. Administrative
2. Course Overview
3. Motivation for PDEs
4. ODEs Recap
5. General introduction to PDEs
6. Tips for Exercise 1

Outline

About Me

Zhenrong Lang (David)
5th semester at D-ITET.

Exactly one year ago, I took this course and wrote the exam.

I am generally interested in Mathematics, Computer Science and Engineering.

For any questions or suggestions to the exercise session, please reach out under:

`zhlang@student.ethz.ch`

The polybox link for this semester:

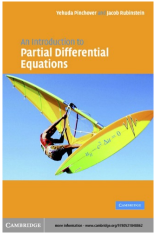
`https://www.polybox.ethz.ch/index.php/s/Elu2hUEoX3pXzgV`

Course Materials

Course Website:

`https://metaphor.ethz.ch/x/2022/hs/401-0353-00L/`

Additional reading:



There is no bonus for this course.
However, the Series each week are concise and helpful for understanding the materials.

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

Outline

Motivation for PDEs

Maxwell's Equations

$$\nabla \cdot \vec{D}(\vec{r},t) = \rho_0(\vec{r},t)$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$$

$$\nabla \times \vec{H}(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}(r,t) + \vec{j_0}(\vec{r},t)$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0$$

The Schrödinger Equation

$$[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})]\Psi(\vec{x},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t)$$

Outline

Equations

Polynomial Equations

$$x^2 - 4x + 3 = 0$$

Solution:

$$x_1 = 1, x_2 = 3$$

Ordinary Differential Equations

$$\begin{cases} \ddot{y}(t) - \dot{y}(t) - 6y(t) = 0 \\ y(0) = 1 \\ \dot{y}(0) = 2 \end{cases}$$

Solution:

$$y(t) = \frac{1}{5}e^{-2t} + \frac{4}{5}e^{3t}$$

Partial Differential Equations

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \\ u(x, t = 0) = sin(x) \\ u_t(x, t = 0) = 0 \end{cases}$$

Solution:

$$u(x,t) = \frac{sin(x + ct) + sin(x - ct)}{2}$$

Definitions for ODEs

General Definition

$$F(t, x, x', \dots, x^{(n-1)}) = x^{(n)}$$

1. order:
   The highest derivative in the equation.
   In this case: n.
2. linear:
   y and its derivatives do not appear as powers or in other functions, but always directly as a summand, that is at most multiplied by a factor, i.e. can be written in the following form:
   $x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \dots + a_1(t)\dot{x}(t) + a_0(t)x(t) = s(t)$
3. homogeneous:
   $s(t) = 0$
4. constant variables
   If the all the coefficients are constant.

Ansatz 1: Exponential Ansatz

Illustration via example: $y''(x) + 2y'(x) + 2y(x) = e^{3x}$

Ansatz 1: Exponential Function
Special case

If the characteristic polynomial has the form: $\lambda^2 + \lambda = 0$. Then the general solution for the ODE is

$$y(x) = \begin{cases} \alpha \cosh(\sqrt{-\lambda} x) + \beta \sinh(\sqrt{-\lambda} x), & \lambda < 0 \\ \alpha + \beta x, & \lambda = 0 \\ \alpha \cos(\sqrt{\lambda} x) + \beta \sin(\sqrt{\lambda} x), & \lambda > 0 \end{cases}$$

Ansatz 2: Method of Integrating Factors

Illustration via example: $x'(t) + \lambda x(t) = 1, x(0) = x_0.$

Ansatz 3: Separation of Variables

Illustration via example: $y'(t) = y^2(t)t + t$

Outline

General introduction to PDEs

Definition

A partial differential equation, or PDE, is an equation involving an unknown function of more than one variable and certain of its partial derivatives.
The general form of a PDE for a function $u(x_1, x_2, ..., x_n)$ is $F(x_1, x_2, ..., u, u_{x_1}, u_{x_2}, ..., u_{x_{11}}) = 0$

1. Order: highest order partial derivative of the function with respect to any variable.
2. Linearity:
3. Quasi-linearity:linear with respect to the highest order derivative
4. homogeneous: every term that doesn't depend on u is equal to 0.

Exercises on classification

1. $uu_x + x^2 u_y = u^3$
2. $uu_x + u^2 u_y + u = e^4$
3. $u_x u_y = x^2 y$
4. $u_t = u_y u_{xx} + u^2 u_{yy} + u_x^2$
5. $u_{xy}^2 = xu + u_y$
6. $u_x + u_{tt} + 2u = e^x$
7. $u_x + u_{tt} + 3uu_x = e^x$
8. $u_{xx} + u_t^2 + e^u = 0$
9. $u_{xx}^2 + u_t + e^u = 0$
10. $usin(u_x) + u_{yy} = 0$

Initial and Boundary Conditions

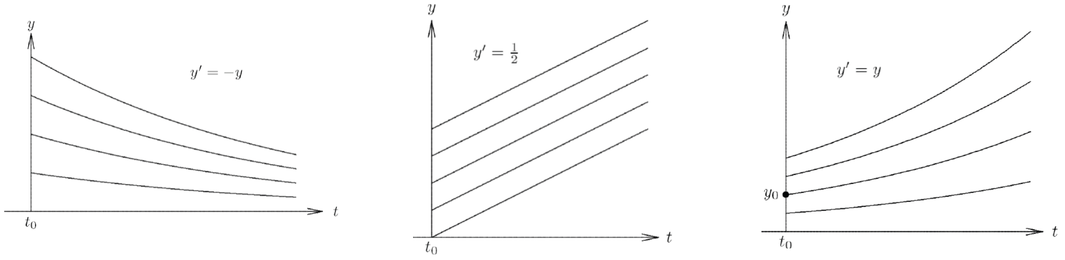
Why do we need initial and boundary conditions?
Example: $u_t = u_{xx}$

Initial Conditions:

Boundary Conditions:

Well-posed Problem

Existence + Uniqueness + Stability



Outline

Tips for Exercise 1

1. Classification of PDEs
2. Solutions to ODEs (Important for later)
   - (a): Separation of Variables
   - (b): particular solution or Method of Integrating factors
   - (c): as (b)
   - (d): as (b)
   - (e): as (b)
   - (f): Exponential Ansatz, case studies on λ
3. Nonexistence of solutions
   - Try to find u_{xy} and u_{yx}. What do you notice?
4. Existence of infinite solutions
   - Try polynomials.
5. Multiple Choice
   - (c) If $v = e^u$, how to express u in terms of v?

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Peers found useful:

1. Introduction to PDEs:
   3Blue1Brown's Playlist on Differential Equations
   https://youtube.com/playlist?list=PLZHQObOWTQDNPOjrT6KVlfJuKtYTftqH6
2. More on ODEs:
   "Ordinary Differential Equations" by Vladimir I. Arnol'd

Additional reading:

1. Chapter 1.5 in the lecture notes: Modelling a stock market

References:

1. Lecture notes on the course website.
2. "An Introduction to Partial Differential Equations" by Yehuda Pinchover and Jacob Rubinstein
3. Princeton University Lecture notes