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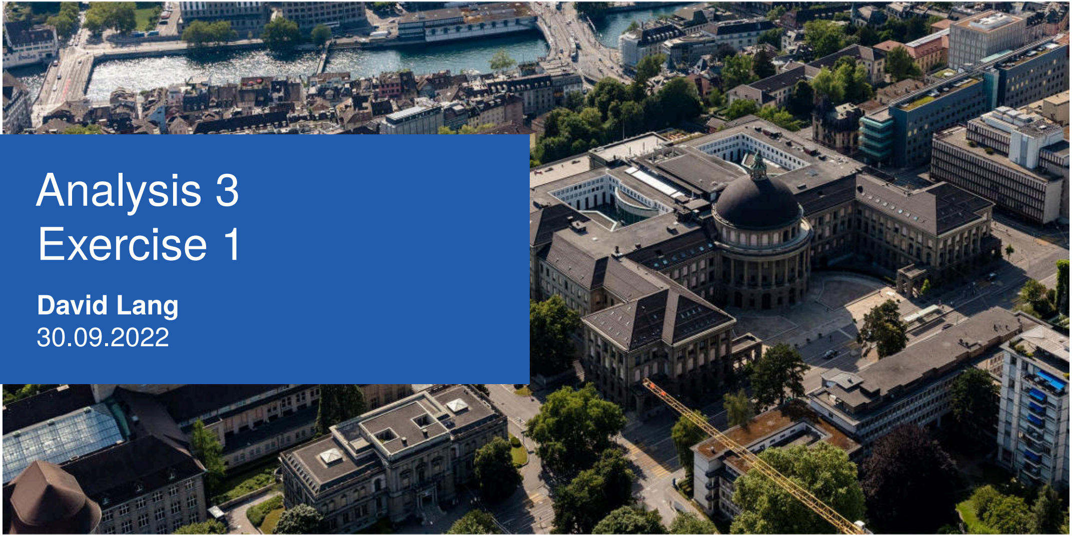

Analysis 3
Exercise 1

David Lang

30.09.2022

Outline

1. Administrative
2. Course Overview
3. Motivation for PDEs
4. ODEs Recap
5. General introduction to PDEs
6. Tips for Exercise 1

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About Me

Zhenrong Lang (David)
5th semester at D-ITET.

Exactly one year ago, I took this course and wrote the exam.

I am generally interested in Mathematics, Computer Science and Engineering.

For any questions or suggestions to the exercise session, please reach out under:

`zhlang@student.ethz.ch`

The polybox link for this semester:

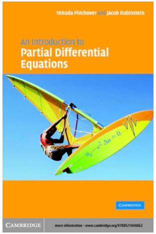
`https://www.polybox.ethz.ch/index.php/s/Elu2hUEoX3pXzgV`

Course Materials

Course Website:

`https://metaphor.ethz.ch/x/2022/hs/401-0353-00L/`

Additional reading:



There is no bonus for this course.
However, the Series each week are concise and helpful for understanding the materials.

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - Laplace equation
    - Maximum principle
    - Separation of variables

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Motivation for PDEs

Maxwell's Equations

$$\nabla \cdot \vec{D}(\vec{r},t) = \rho_0(\vec{r},t)$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$$

$$\nabla \times \vec{H}(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}(r,t) + \vec{j_0}(\vec{r},t)$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0$$

solution is a vector field

The Schrödinger Equation

$$[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})]\Psi(\vec{x},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t)$$

solution is a scalar multivariable function

we will focus on these equations in Analysis 3

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Equations

Polynomial Equations

$$x^2 - 4x + 3 = 0$$

Solution:

$$x_1 = 1, x_2 = 3$$

Ordinary Differential Equations

$$\begin{cases} \ddot{y}(t) - \dot{y}(t) - 6y(t) = 0 \\ y(0) = 1 \\ \dot{y}(0) = 2 \end{cases}$$

Solution:

$$y(t) = \frac{1}{5}e^{-2t} + \frac{4}{5}e^{3t}$$

2nd order, linear, homogeneous ODE
solution space: 2-dimensional $\mathbb{R}$-Vectorspace
General Solution: linear combinations of two linearly independent functions

Partial Differential Equations

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \\ u(x, t=0) = sin(x) \\ u_t(x, t=0) = 0 \end{cases}$$

Solution:

$$u(x,t) = \frac{sin(x+ct) + sin(x-ct)}{2}$$

solution is a function with more than one independent variable

Definitions for ODEs

General Definition

$$F(t, x, x', \dots, x^{(n-1)}) = x^{(n)}$$

1. order:
   The highest derivative in the equation.
   In this case: n.
2. linear:
   y and its derivatives do not appear as powers or in other functions, but always directly as a summand, that is at most multiplied by a factor, i.e. can be written in the following form:
   $x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \dots + a_1(t)\dot{x}(t) + a_0(t)x(t) = s(t)$
3. homogeneous:
   $s(t) = 0$
4. constant variables
   If the all the coefficients are constant.

$L[u(t)] = s(t)$ $L[v(t)] = s(t)$

$L[u-v] \overset{\text{if linear}}{=} L[u] - L[v] = s(t) - s(t) = 0$

$$\begin{bmatrix} 1 \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} \cdot \begin{bmatrix} x^{(n)}(t) \\ x^{(n-1)}(t) \\ \vdots \\ \dot{x}(t) \\ x(t) \end{bmatrix} = s(t)$$

$x(t) \cdot x'(t) = s(t)$ ← not linear

Ansatz 1: Exponential Ansatz

Illustration via example: $y''(x) + 2y'(x) + 2y(x) = e^{3x}$ (linear, 2nd order, inhomogeneous)

solution = homogeneous + particular

homogeneous:

Ansatz $y_h(x) = e^{\lambda x}$

$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 2 e^{\lambda x} = 0$

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda = -1 \pm i$

$y_h(x) = e^{-x}(C_1 \cos(x) + C_2 \sin(x))$

particular:

$S(x) = e^{3x}$

Ansatz: $y_p(x) = A_0 e^{3x}$

$9A_0 e^{3x} + 6A_0 e^{3x} + 2A_0 e^{3x} = e^{3x}$

$A_0 = \frac{1}{17}$

$y_p(x) = \frac{1}{17} e^{3x}$

$$y(t) = e^{-x}(C_1 \cos(x) + C_2 \sin(x)) + \frac{1}{17} e^{3x}$$

need two linearly independent initial values to find out C_1 & C_2

$$\begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \; \\ \; \end{bmatrix}$$

Ansatz 1: Exponential Function

Special case

If the characteristic polynomial has the form: $\lambda^2 + \lambda = 0$. Then the general solution for the ODE is

$$y(x) = \begin{cases} \alpha \cosh(\sqrt{-\lambda}x) + \beta \sinh(\sqrt{-\lambda}x), & \lambda < 0 \\ \alpha + \beta x, & \lambda = 0 \\ \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x), & \lambda > 0 \end{cases}$$

Ansatz 2: Method of Integrating Factors

Illustration via example: $x'(t) + \lambda x(t) = 1, x(0) = x_0.$

$$e^{\lambda t} x'(t) + e^{\lambda t} \cdot \lambda x(t) = e^{\lambda t}$$

$$\frac{d}{dt}\left(e^{\lambda t} x(t)\right) = e^{\lambda t}$$

$$e^{\lambda t} x(t) = \int e^{\lambda t}\, dt$$

$$e^{\lambda t} x(t) = \frac{1}{\lambda} e^{\lambda t} + C$$

$$x(t) = \frac{1}{\lambda} + C e^{-\lambda t}$$

$$x(0) = x_0 \rightarrow C = x_0 - \frac{1}{\lambda}$$

$$x(t) = \frac{1}{\lambda} + \left(x_0 - \frac{1}{\lambda}\right) e^{-\lambda t}$$

$$x'(t) + 4x(t) = e^{-t} \qquad x(0) = \frac{4}{3}$$

$$e^{4t} x'(t) + e^{4t} 4x(t) = e^{3t}$$

$$\left(e^{4t} x(t)\right)' = e^{3t}$$

$$e^{4t} x(t) = \int e^{3t}\, dt$$

$$e^{4t} x(t) = \frac{1}{3} e^{3t} + C$$

$$x(t) = e^{-4t}\left[\frac{1}{3} e^{3t} + C\right] = \frac{1}{3} e^{-t} + C e^{-4t}$$

$$x(0) = \frac{4}{3} \rightarrow C = 1$$

$$x(t) = \frac{1}{3} e^{-t} + e^{-4t}$$

In general: $x'(t) + p(t) x(t) = f(t)$

$$r(t) x'(t) + r(t) p(t) x(t) = r(t) f(t) \qquad r(t) = e^{\int p(t)\, dt}$$

$$\frac{d}{dt}\left[r(t) x(t)\right] = r(t) f(t)$$

$$x(t) = \frac{1}{r(t)} \int r(t) f(t)\, dt$$

Ansatz 3: Separation of Variables

Illustration via example: $y'(t) = y^2(t)t + t$

$$\frac{dy}{dt} = t(y^2+1)$$

$$\frac{dy}{y^2+1} = t\,dt$$

$$\int \frac{dy}{y^2+1} = \int t\,dt$$

$$\arctan(y) = \frac{1}{2}t^2 + C$$

$$y(t) = \tan\left(\frac{1}{2}t^2 + C\right)$$

$$t^3 x'(t) + 4t^2 x(t) = 0$$

$$x'(t) = -\frac{4}{t}x(t)$$

$$\frac{dx}{dt} = -\frac{4}{t}x(t)$$

$$\frac{dx}{x} = -\frac{4}{t}dt$$

$$\int \frac{dx}{x} = \int -\frac{4}{t}dt$$

$$\ln|x| = \ln|t|^{-4} + C$$

$$x(t) = D \cdot t^{-4}$$

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General introduction to PDEs

Definition

A partial differential equation, or PDE, is an equation involving an unknown function of more than one variable and certain of its partial derivatives.
The general form of a PDE for a function $u(x_1, x_2, ..., x_n)$ is $F(x_1, x_2, ..., u, u_{x_1}, u_{x_2}, ..., u_{x_{11}}) = 0$

1. Order: highest order partial derivative of the function with respect to any variable.
2. Linearity: $\begin{bmatrix} \vdots \end{bmatrix} \cdot \begin{bmatrix} u \\ u_x \\ u_y \\ \vdots \\ u_{\dots} \end{bmatrix} = \dots$ i.e. $\begin{bmatrix} t \\ t^2 \\ x \end{bmatrix} \cdot \begin{bmatrix} u \\ u_t \\ u_x \end{bmatrix} = 0$ $tu + t^2 u_t + x u_x = 0$ is linear

   $u_t + u u_x = 0$ is not linear, i.e. can't be written in such form
3. Quasi-linearity:linear with respect to the highest order derivative

   subset of non-linear PDE $\begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{yy} \\ \dots \end{bmatrix} = \dots$ $u_x^2 + u_y^2 + u_{xy} + u_{yy} = 0$ is quasilinear

   $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} u_{xy} \\ u_{yy} \end{bmatrix} + u_x^2 + u_y^2 = 0$
4. homogeneous: every term that doesn't depend on u is equal to 0.

Exercises on classification

		linear?	order?	quasi-linear?
1.	$uu_x + x^2u_y = u^3$	✗	1	✓
2.	$uu_x + u^2u_y + u = e^4$	✗	1	✓
3.	$u_xu_y = x^2y$	✗	1	✗
4.	$u_t = u_yu_{xx} + u^2u_{yy} + u_x^2$	✗	2	✓
5.	$u_{xy}^2 = xu + u_y$	✗	2	✗
6.	$u_x + u_{tt} + 2u = e^x$	✓	2	
7.	$u_x + u_{tt} + 3uu_x = e^x$	✗	2	✓
8.	$u_{xx} + u_t^2 + e^u = 0$	✗	2	✓
9.	$u_{xx}^2 + u_t + e^u = 0$	✗	2	✗
10.	$usin(u_x) + u_{yy} = 0$	✗	2	✓

Initial and Boundary Conditions

Why do we need initial and boundary conditions?

Example: $u_t = u_{xx}$ heat equation

$u(x,t) = \frac{1}{2}x^2 + t$

$u(x,t) = e^x e^t$

both valid solutions but not physical

Initial Conditions:

$u(x,t_0) = f(x)$

$u_t(x,t_0) = g(x)$

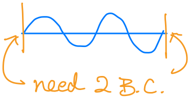
initial conditions = highest order of time derivative

Boundary Conditions:

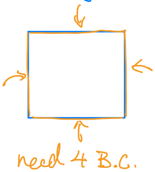
$u(x=0,t) = f(t)$

$u_x(x=0,t) = g(t)$

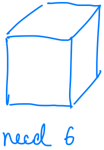
$u_{xx} = u_{tt}$



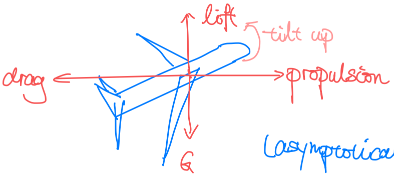
$u_{xx} + u_{yy} = u_{tt}$



$u_{xx} + u_{yy} + u_{zz} = u_{tt}$



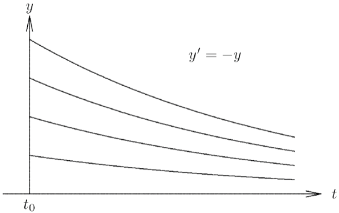
Well-posed Problem



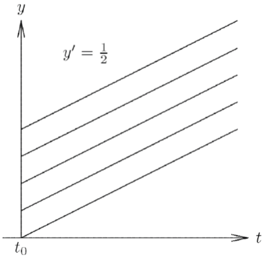
(asymptotically) stable → no crash

Existence + Uniqueness + Stability

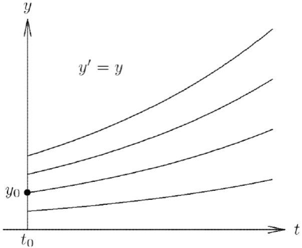
The graphs are just for intuition/illustration



asymptotically stable



stable



unstable

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Tips for Exercise 1

1. Classification of PDEs
2. Solutions to ODEs (Important for later)
   - (a): Separation of Variables
   - (b): particular solution or Method of Integrating factors
   - (c): as (b)
   - (d): as (b)
   - (e): as (b)
   - (f): Exponential Ansatz, case studies on λ
3. Nonexistence of solutions
   - Try to find u_{xy} and u_{yx}. What do you notice?
4. Existence of infinite solutions
   - Try polynomials.
5. Multiple Choice
   - (c) If $v = e^u$, how to express u in terms of v?

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Peers found useful:

1. Introduction to PDEs:
   3Blue1Brown's Playlist on Differential Equations
   https://youtube.com/playlist?list=PLZHQObOWTQDNPOjrT6KVlfJuKtYTftqH6
2. More on ODEs:
   "Ordinary Differential Equations" by Vladimir I. Arnol'd

Additional reading:

1. Chapter 1.5 in the lecture notes: Modelling a stock market

References:

1. Lecture notes on the course website.
2. "An Introduction to Partial Differential Equations" by Yehuda Pinchover and Jacob Rubinstein
3. Princeton University Lecture notes