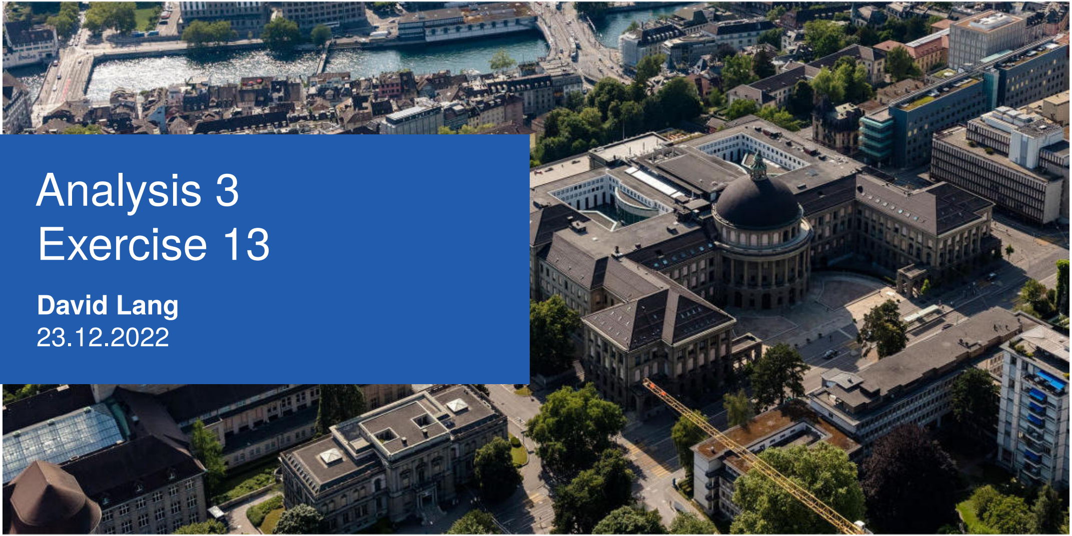

Analysis 3
Exercise 13

David Lang

23.12.2022

Outline

1. Serie 12 Review
2. Course Overview
3. Laplace's equation in circular domains
4. Kahoot!
5. Tips for Serie 13

Outline

Serie 12 Review

1. Necessary condition
2. Separation of variables
3. Neumann problem

   $$\sin(y) - \frac{2}{\pi} = a_0 + \sum_{n=1}^{\infty} a_n \cos(ny) \text{ for } y \in [0, \pi]$$

   extend the function evenly: $|\sin(y)| - \frac{2}{\pi}$ for $y \in \mathbb{R},\ a_0 = 0, a_n = -\frac{4}{\pi}\frac{1}{(n-1)(n+1)},\ n = 2, 4, 6, ...$

   $$\sin(y) - \frac{2}{\pi} = -\frac{4}{\pi}\left[\frac{\cos(2y)}{1\cdot 3} + \frac{\cos(4y)}{3\cdot 5} + \frac{\cos(6y)}{5\cdot 7} + ...\right] \text{ for } y \in [0, \pi]$$

4. Laplace operator and rotations

   $v(x, y)$ the total differential for $v:\ dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

   $x = \cos(\theta)s + \sin(\theta)t,\ y = -\sin(\theta)s + \cos(\theta)t$

   $$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s}$$

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - **Laplace equation**
    - Maximum principle
    - **Separation of variables**

Outline

Laplace's equation on a disk

Domain: $D = \{(r, \theta) \mid 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi\}$

$$\begin{cases} \Delta w(r,\theta) = 0 & 0 \leq r < a,\ 0 \leq \theta \leq 2\pi \\ w(a,\theta) = g(\theta) & 0 \leq \theta \leq 2\pi \end{cases}$$

periodicity: $\Theta(0) = \Theta(2\pi),\ \Theta'(0) = \Theta'(2\pi)$

General Solution:

$$w(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

Laplace's equation on a ring

Domain: $D = \{(r, \theta) \mid a \leq r \leq b,\ 0 \leq \theta \leq 2\pi\}$

$$\begin{cases} \Delta w(r,\theta) = 0 & a < r < b,\ 0 \leq \theta \leq 2\pi \\ w(a,\theta) = h(\theta) & 0 \leq \theta \leq 2\pi \\ w(b,\theta) = g(\theta) & 0 \leq \theta \leq 2\pi \end{cases}$$

periodicity: $\Theta(0) = \Theta(2\pi),\ \Theta'(0) = \Theta'(2\pi)$

General Solution:

$$w(r,\theta) = E + F\ln(r) + \sum_{n=1}^{\infty} \left[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) + C_n r^{-n} \cos(n\theta) + D_n r^{-n} \sin(n\theta)\right]$$

Example 1
Laplace equation on a ring

$$\begin{cases} \Delta u = 0 & 2 < r < 4, \ -\pi < \theta \leq \pi \\ u(2,\theta) = 0, & -\pi < \theta \leq \pi \\ u(4,\theta) = \sin(\theta), & -\pi < \theta \leq \pi \end{cases}$$

Example 1

Laplace equation on a ring

$$\begin{cases} \Delta u = 0 & 2 < r < 4, \ -\pi < \theta \leq \pi \\ u(2,\theta) = 0, & -\pi < \theta \leq \pi \\ u(4,\theta) = \sin(\theta), & -\pi < \theta \leq \pi \end{cases}$$

Laplace's equation on a wedge

Domain: $D = \{(r,\theta) \mid 0 \le r \le a,\ 0 \le \theta \le \gamma\}$

$$\begin{cases} \Delta w(r,\theta) = 0 & 0 \le r < a,\ 0 \le \theta \le \gamma \\ w(r,0) = 0 & 0 \le r < a \\ w(r,\gamma) = 0 & 0 \le r < a \\ w(a,\theta) = g(\theta) & 0 \le \theta \le \gamma \end{cases}$$

General Solution:

$$w(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}$$

Laplace's equation on an annulus sector

Domain: $D = \{(r,\theta) \mid a \le r \le b,\ 0 \le \theta \le \gamma\}$

$$\begin{cases} \Delta w(r,\theta) = 0 & a < r < b,\ 0 < \theta < \gamma \\ w(r,0) = 0 & a < r < b \\ w(r,\gamma) = 0 & a < r < b \\ w(a,\theta) = g(\theta) & 0 \le \theta \le \gamma \\ w(b,\theta) = h(\theta) & 0 \le \theta \le \gamma \end{cases}$$

General Solution:

$$w(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}} + B_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{-\frac{n\pi}{\gamma}}$$

Example 2
Laplace's equation on an annulus sector

$$\begin{cases} \Delta w = 0, & a < r < b,\ \alpha < \theta < \beta \\ w(r,\alpha) = 0, & a < r < b \\ w(r,\beta) = 0, & a < r < b \\ w(a,\theta) = g(\theta), & \alpha \le \theta \le \beta \\ w(b,\theta) = h(\theta), & \alpha \le \theta \le \beta \end{cases}$$

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Tips for Serie 13

1. Harmonic function in the disk
   - $(x, y) = (r\cos(\theta), r\sin(\theta))$
   - $\cos(\theta)^3 = \frac{1}{4}(3\cos(\theta) + \cos(3\theta))$
2. Harmonic function in the annulus
   - See example 2
3. Big on the boundary, small inside
   - The general solution is
   - $w(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$
   - How does n affect the value?

ETH zürich

Self-promotion:

Teaching Assistant for ***Introduction to Machine Learning*** from D-INFK next semester

Instructor: Prof. Dr. Andreas Krause and Prof. Dr. Fan Yang

The course introduces the foundations of learning and making predictions from data.

This concludes the exercises for this semester.

Merry Christmas and Happy New Year!

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein