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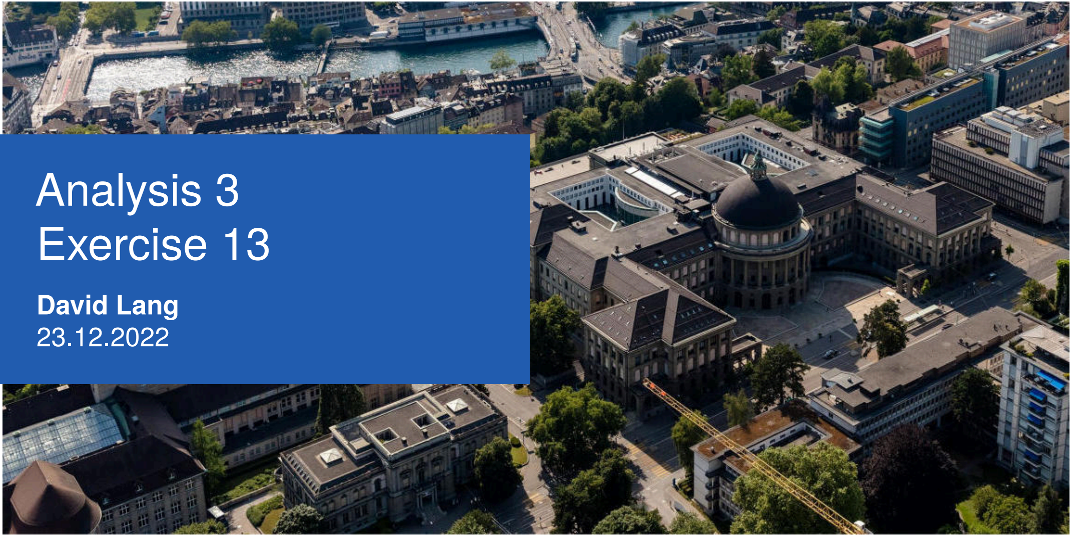

# Analysis 3
# Exercise 13

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23.12.2022

# Outline

1. Serie 12 Review
2. Course Overview
3. Laplace's equation in circular domains
4. Kahoot!
5. Tips for Serie 13

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# Serie 12 Review

1. Necessary condition
2. Separation of variables
3. Neumann problem

$$\sin(y) - \frac{2}{\pi} = a_0 + \sum_{n=1}^{\infty} a_n \cos(ny) \text{ for } y \in [0, \pi]$$

extend the function evenly: $|\sin(y)| - \frac{2}{\pi}$ for $y \in \mathbb{R}$, $a_0 = 0, a_n = -\frac{4}{\pi}\frac{1}{(n-1)(n+1)}, \; n = 2, 4, 6, \ldots$

$$\sin(y) - \frac{2}{\pi} = -\frac{4}{\pi}\left[\frac{\cos(2y)}{1 \cdot 3} + \frac{\cos(4y)}{3 \cdot 5} + \frac{\cos(6y)}{5 \cdot 7} + \ldots\right] \text{ for } y \in [0, \pi]$$

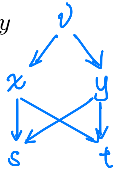
4. Laplace operator and rotations

chain rule

$v(x, y)$ the total differential for $v$ : $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$x = \cos(\theta)s + \sin(\theta)t, \; y = -\sin(\theta)s + \cos(\theta)t$

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} = v_x \cdot \cos(\theta) - v_y \sin(\theta)$$




without chain rule

$dx = \cos(\theta)\,ds + \sin(\theta)\,dt$

$dy = -\sin(\theta)\,ds + \cos(\theta)\,dt$

$dv = v_x \cdot (\cos(\theta)\,ds + \sin(\theta)\,dt) + v_y \cdot (-\sin(\theta)\,ds + \cos(\theta)\,dt)$

$= (v_x \cdot \cos(\theta) - v_y \sin(\theta))\,ds + (v_x \sin(\theta) + v_y \cos(\theta))\,dt$

# Outline

# Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - **Laplace equation**
    - Maximum principle
    - **Separation of variables**

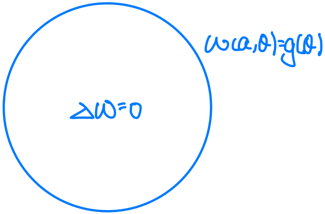
# Outline

# Laplace's equation on a disk

"Pizza fresh out of oven"

Domain: $D = \{(r, \theta) \mid 0 \leq r \leq a,\ 0 \leq \theta \leq 2\pi\}$

$$\begin{cases} \Delta w(r,\theta) = 0 & 0 \leq r < a,\ 0 \leq \theta \leq 2\pi \\ w(a,\theta) = g(\theta) & 0 \leq \theta \leq 2\pi \end{cases}$$



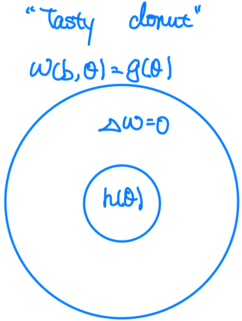

periodicity: $\Theta(0) = \Theta(2\pi),\ \Theta'(0) = \Theta'(2\pi)$

General Solution:

$$w(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

# Laplace's equation on a ring

Domain: $D = \{(r,\theta) \mid a \le r \le b,\ 0 \le \theta \le 2\pi\}$




$$\begin{cases} \Delta w(r,\theta) = 0 & a < r < b,\ 0 \le \theta \le 2\pi \\ w(a,\theta) = h(\theta) & 0 \le \theta \le 2\pi \\ w(b,\theta) = g(\theta) & 0 \le \theta \le 2\pi \end{cases}$$

periodicity: $\Theta(0) = \Theta(2\pi),\ \Theta'(0) = \Theta'(2\pi)$

General Solution:

$$w(r,\theta) = E + F\ln(r) + \sum_{n=1}^{\infty} \left[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) + C_n r^{-n} \cos(n\theta) + D_n r^{-n} \sin(n\theta)\right]$$

# Example 1

## Laplace equation on a ring

$$\begin{cases} \Delta u = 0 & 2 < r < 4, \quad -\pi < \theta \leq \pi \\ u(2,\theta) = 0, & -\pi < \theta \leq \pi \\ u(4,\theta) = \sin(\theta), & -\pi < \theta \leq \pi \end{cases}$$

$$u(r,\theta) = E + F\ln(r) + \sum_{n=1}^{\infty} [A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) + C_n r^{-n} \cos(n\theta) + D_n r^{-n} \sin(n\theta)]$$

$$u(2,\theta) = E + F\ln(2) + \sum_{n=1}^{\infty} A_n 2^n \cos(n\theta) + B_n 2^n \sin(n\theta) + C_n 2^{-n} \cos(n\theta) + D_n 2^{-n} \sin(n\theta)] = 0$$

$$u(4,\theta) = E + F\ln(4) + \sum_{n=1}^{\infty} A_n 4^n \cos(n\theta) + B_n 4^n \sin(n\theta) + C_n 4^{-n} \cos(n\theta) + D_n 4^{-n} \sin(n\theta)] = \sin(\theta)$$

$$\begin{cases} E + F\ln(2) = 0 \\ E + F\ln(4) = 0 \end{cases} \qquad \begin{cases} A_n 2^n + C_n 2^{-n} = 0 \\ A_n 4^n + C_n 4^{-n} = 0 \end{cases} \quad n \geq 1 \qquad \begin{cases} B_n 2^n + D_n 2^{-n} = 0 \\ B_n 4^n + D_n 4^{-n} = 0 \end{cases} \quad n \geq 2$$

$$\begin{cases} E = 0 \\ F = 0 \end{cases} \qquad \begin{cases} A_n = 0 \\ C_n = 0 \end{cases} \quad n \geq 1 \qquad \begin{cases} B_n = 0 \\ D_n = 0 \end{cases} \quad n \geq 2$$

# Example 1

## Laplace equation on a ring

$$\begin{cases} \Delta u = 0 & 2 < r < 4, \quad -\pi < \theta \leq \pi \\ u(2,\theta) = 0, & -\pi < \theta \leq \pi \\ u(4,\theta) = \sin(\theta), & -\pi < \theta \leq \pi \end{cases}$$

$$\begin{cases} 2B_1 + \frac{1}{2}D_1 = 0 \\ 4B_1 + \frac{1}{4}D_1 = 1 \end{cases} \qquad \begin{cases} B_1 = \frac{1}{3} \\ D_1 = -\frac{4}{3} \end{cases}$$

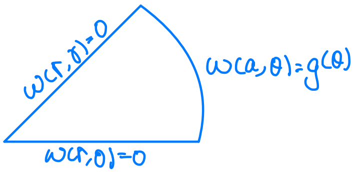
$$u(r,\theta) = r\frac{\sin(\theta)}{3} - \frac{4\sin(\theta)}{3r}$$

# Laplace's equation on a wedge

Domain: $D = \{(r, \theta) \mid 0 \leq r \leq a,\ 0 \leq \theta \leq \gamma\}$

"Pizza slice before the first bite"

$$\begin{cases} \Delta w(r,\theta) = 0 & 0 \leq r < a,\ 0 \leq \theta \leq \gamma \\ w(r,0) = 0 & 0 \leq r < a \\ w(r,\gamma) = 0 & 0 \leq r < a \\ w(a,\theta) = g(\theta) & 0 \leq \theta \leq \gamma \end{cases}$$




General Solution:

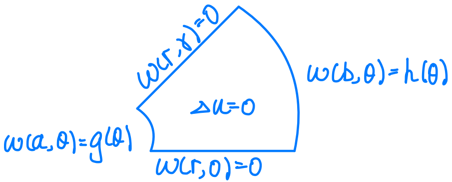
$$w(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}$$

# Laplace's equation on an annulus sector

Domain: $D = \{(r, \theta) \mid a \leq r \leq b,\ 0 \leq \theta \leq \gamma\}$

"Pizza slice with first bite gone"

$$\begin{cases} \Delta w(r,\theta) = 0 & a < r < b,\ 0 < \theta < \gamma \\ w(r,0) = 0 & a < r < b \\ w(r,\gamma) = 0 & a < r < b \\ w(a,\theta) = g(\theta) & 0 \leq \theta \leq \gamma \\ w(b,\theta) = h(\theta) & 0 \leq \theta \leq \gamma \end{cases}$$




General Solution:

$$w(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}} + B_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{-\frac{n\pi}{\gamma}}$$

# Example 2

Laplace's equation on an annulus sector

$$\begin{cases} \Delta w = 0, & a < r < b,\ \alpha < \theta < \beta \\ w(r,\alpha) = 0, & a < r < b \\ w(r,\beta) = 0, & a < r < b \\ w(a,\theta) = g(\theta), & \alpha \le \theta \le \beta \\ w(b,\theta) = h(\theta), & \alpha \le \theta \le \beta \end{cases}$$

$$\begin{cases} \Delta v = 0 & a<r<b,\ 0<\theta<\beta-\alpha \\ v(r,0) = 0 & a<r<b \\ v(r,\beta-\alpha)=0 & a<r<b \\ v(a,\theta) = g(\theta+\alpha) & 0<\theta<\beta-\alpha \\ v(b,\theta) = h(\theta+\alpha) & 0<\theta<\beta-\alpha \end{cases}$$

Change of variables: $v(r,\theta) = w(r,\theta+\alpha)$

Laplace operator is rotational invariant (Serie 12.4)

General solution:

$$v(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\beta-\alpha}\theta\right) r^{\frac{n\pi}{\beta-\alpha}} + B_n \sin\left(\frac{n\pi}{\beta-\alpha}\theta\right) r^{-\frac{n\pi}{\beta-\alpha}}$$

change back:

$$w(r,\theta) = v(r,\theta-\alpha) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right) r^{\frac{n\pi}{\beta-\alpha}} + B_n \sin\left(\frac{n\pi}{\beta-\alpha}(\theta-\alpha)\right) r^{-\frac{n\pi}{\beta-\alpha}}$$

# Outline

# Kahoot!
## Question 1

Classify the following PDE:

$$e^{\Delta u} = u$$

- (a) linear and homogeneous
- (b) linear and inhomogeneous
- **(c) not linear, but quasilinear**
- (d) fully nonlinear

The PDE can be rewritten as $\Delta u = \ln(u)$.
This is second order, not linear, but quasilinear.

# Kahoot!
## Question 2

Classify the following PDE:

$$u_{xx}^2 + u_t + e^u = 0$$

- (a) linear and homogeneous
- (b) linear and inhomogeneous
- (c) not linear, but quasilinear
- **(d) fully nonlinear**

The PDE is of the 2nd order and fully nonlinear, because $u_{xx}$ appears nonlinearly.

# Kahoot!
## Question 3

A solution must be at least $C^2$ in order for the problem to be well-posed.

- (a) True
- **(b) False**

The solution to a well-posed problem should exist, be unique and stable.
The solution does not have to be strong (classic) (i.e., all the derivatives of the solution that appear in the PDE exist and are continuous).
Weak solutions (the ones we have encountered studying conservation laws) are also valid solutions.

# Kahoot!

## Question 4

Consider a quasilinear PDE of the form $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$.
The geometrical interpretation of the method of characteristics that:

- (a) $(a,b,c)$ is normal to the surface $u(x,y)$
- (b) $(a,b,c)$ is orthogonal to the initial curve
- **(c) $(a,b,c)$ is orthogonal to $(u_x,u_y,-1)$**
- (d) the characteristic curves are parallel to each other.

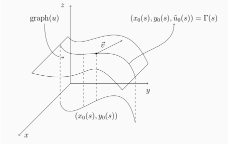
$(a,b,c)$ is tangent to the surface $u(x,y)$.
Orthogonality is not required.
The equation can be rewritten as $(a,b,c)\cdot(u_x,u_y,-1)=0$, which means that the vector $(a,b,c)$ must be orthogonal to the normal vector.
(d) does not hold in general.

# Kahoot!

## Question 5

Which of the following statements is true?

- (a) Method of characteristics guarantees global existence and uniqueness
- **(b) First-odrer PDEs relate the solution surface to its tangent plane**
- (c) The parameterization of the initial curve is unique.
- (d) The characteristic curves and the initial curve never cross each other.




If the transversality condition holds, there exists a unique local solution, no conclusion can be drawn for global existence and uniqueness.
The initial curve is a parametrization of the initial conditions, and it is not unique.
The characteristic curves span the solution space from the initial curve.

# Kahoot!
## Question 6

The initial condition is given as $u(1, y) = f(y)$,
one correct parametrization could be

- (a) $(s, 1, f(s))$
- (b) $(s, s, f(s))$
- **(c)** $(1, s, f(s))$
- (d) none of the above

$x$ is kept at constant 1.

# Kahoot!
## Question 7

If the transversality condition holds at $(0, s_0)$, then

- (a) there exist multiple solutions in a neighborhood of $(x_0(s_0), y_0(s_0))$
- (b) there exist multiple global solutions
- **(c) there exists a unique solution in a neighborhood of $(x_0(s_0), y_0(s_0))$**
- (d) there exists a unique global solution

$$\det \begin{pmatrix} \frac{\partial x}{\partial t}(0, s_0) & \frac{\partial y}{\partial t}(0, s_0) \\ \frac{\partial x}{\partial s}(0, s_0) & \frac{\partial y}{\partial s}(0, s_0) \end{pmatrix} \neq 0$$

This is exactly the existence and uniqueness theorem.

# Kahoot!
## Question 8

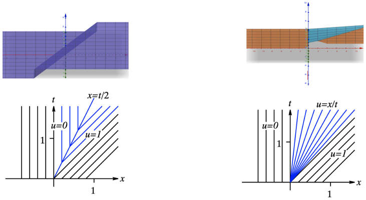
Consider the burgers equation with the following initial data, the critical time $y_c$ is infinite


- (a) True
- **(b) False**

Solutions of conservation laws are constant along their characteristics, which are straight lines. Indeed, for each $s \in \mathbb{R}$, the characteristic through a point $(s, 0)$ is the line in the $(x, y)$-plane going through $(s, 0)$ with slope $1/c(u_0(s))$ and on this line $u$ is equal to the constant $u_0(s)$.
If $c(u_0(s))s < 0$, then there exists a time when the characteristics cross. Heuristically you can think about the latter condition as when a faster characteristic starts from a point behind a slower characteristic.

# Kahoot!
## Question 9

Which of the following solutions satisfies the entropy condition?




- (a) only left
- **(b) only right**
- (c) none
- (d) both

The emergence of characteristics from a shock is interpreted as the creation of information, which should be forbidden.

# Kahoot!
## Question 10

The classification of linear second-order PDEs is a global property, if

$$\mathcal{L}[u] = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

- (a) $a, b, c$ only depends on $x, y$
- **(b) The PDE has constant coefficients**
- (c) $g = 0$
- (d) The discriminant is bigger than $0$

$$\delta(\mathcal{L})(x_0, y_0) = b^2(x_0, y_0) - a(x_0, y_0)c(x_0, y_0)$$

(a) is always true because the PDE is linear.
(b) is true because the discriminant is the same for all $(x_0, y_0)$
(c) is the condition for a homogeneous 2nd order linear PDE.
(d) is a condition for a hyperbolic PDE.

# Kahoot!
## Question 11

The singularities of the initial condition of the wave equation are smoothed out for $t > 0$

- (a) True
- **(b) False**

The singularities of the solution propagate along the characteristics, which are straight lines.

# Kahoot!
## Question 12

The homogeneous heat equation with Neumann boundary conditions converge to

$$\begin{cases} u_t - u_{xx} = 0 & (x,t) \in (0,L) \times (0,\infty) \\ u_x(0,t) = u_x(L,t) & t > 0 \\ u(x,0) = f(x) & x \in (0,L) \end{cases}$$

- **(a)** $\frac{1}{L} \int_0^L f(x)\, dx$
- (b) $\frac{2}{L} \int_0^L f(x)\, dx$
- (c) $\frac{1}{2L} \int_0^L f(x)\, dx$
- (d) $\frac{L}{2} \int_0^L f(x)\, dx$

The PDE models exactly the temperature of a rod with isolated ends. Intuitively, the temperature converges to the average. The same result could be found by writing down the general formula and letting $t \to \infty$.

# Kahoot!
## Question 13

Laplace's equation does NOT describe which of the following situation:

- **(a) steady-state wave equation**
- (b) steady-state heat equation
- (c) height of a tightly stretched membrane
- (d) electric potential where there is no charge

Even standing waves have time dependencies, so (a) is wrong. The other cases have been covered in previous exercises.

# Kahoot!
## Question 14

The necessary condition for the existence of a solution to the Neumann problem states:
heat influx = heat outflux

$$\begin{cases} \Delta u(x,y) = \rho(x,y) & (x,y) \in D \\ \partial_v u(x,y) = g(x,y) & (x,y) \in D \end{cases}$$

- True
- **False**

The statement would be true for the Laplace equation. However, for the non-homogeneous case (i.e., the Poisson Equation) the necessary condition is $\int_{\partial D} g(x(s), y(s))\, ds = \int_D \rho(x,y)\, dx\, dy$, which relates the heat generated inside the domain with the heat flux on the boundary.

# Kahoot!
## Question 15

If the compatibility condition is fulfilled, how many solutions does the Laplace equation with Neumann boundary condition have?

- one unique
- **infinitly many**
- no
- too little information

If $u(x, y)$ is a solution to the problem, then $u(x, y) + C$ is a solution, where $C$ is an arbitrary constant.

# Outline

# Tips for Serie 13

1. Harmonic function in the disk
   - $(x, y) = (r\cos(\theta), r\sin(\theta))$
   - $\cos(\theta)^3 = \frac{1}{4}(3\cos(\theta) + \cos(3\theta))$
2. Harmonic function in the annulus
   - See example 2
3. Big on the boundary, small inside
   - The general solution is
   - $w(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$
   - How does n affect the value?

ETH zürich

Self-promotion:

Teaching Assistant for ***Introduction to Machine Learning*** from D-INFK next semester

Instructor: Prof. Dr. Andreas Krause and Prof. Dr. Fan Yang

The course introduces the foundations of learning and making predictions from data.

This concludes the exercises for this semester.

Merry Christmas and Happy New Year!

References:


1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein