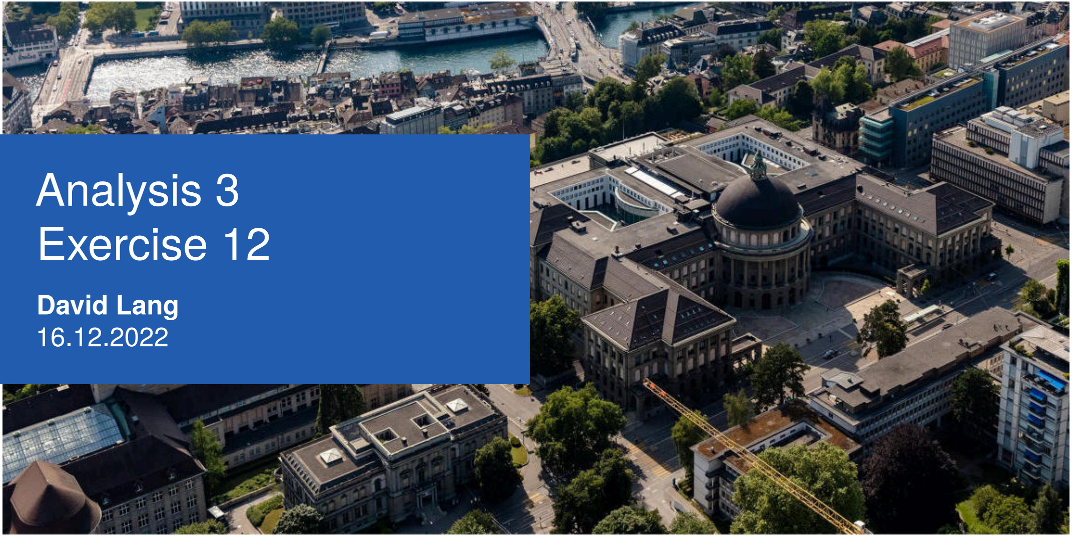

Analysis 3
Exercise 12

David Lang

16.12.2022

Outline

1. Serie 11 Review
2. Course Overview
3. Compatibility condition
4. Laplace's equation on circular domains
5. Tips for Serie 12

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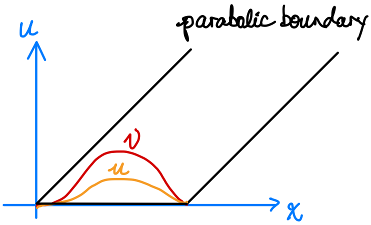
Serie 11 Review



1. Separation of variables for elliptic equations
   - (a)
   - (b)
2. Heat Equation
   - The comparison principle for solutions of the heat equation: If u and v are two solutions of the heat equation, and $u \leq v$ for the initial and boundary condition, then $u \leq v$ everywhere.
   - Prove:
   - Consider $w = u - v$.
   - By linearity w satisfies the heat equation, and $w \leq 0$ on the parabolic boundary.
   - By the weak maximum principle, $w \leq 0$ everywhere, thus $u \leq v$ everywhere.
3. Uniqueness of solutions
   -

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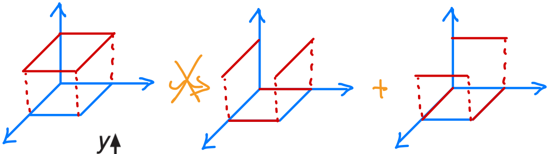
Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - **Laplace equation**
    - Maximum principle
    - **Separation of variables**

Outline

Compatibility condition

Laplace's Equation with Dirichlet boundary condition



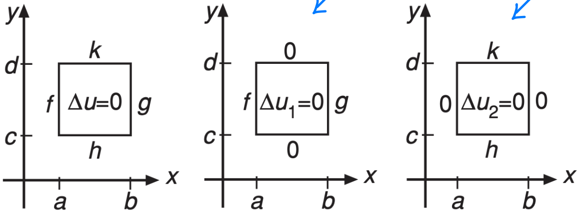


Figure 7.4 Separation of variables in rectangles.

We assumed last time that the compatibility condition holds.

$$f(c) = f(d) = g(c) = g(d) = h(a) = h(b) = k(a) = k(b) = 0$$

The uniqueness theorem guarantees that $u = u_1 + u_2$ is a unique solution.

Compatibility condition
Laplace's Equation with Dirichlet boundary condition

When we split the problem for u into two problems for u_1 and u_2, the boundary data may not be continuous anymore, even if they are continuous in the original problem.

We therefore present a method for transforming a Dirichlet problem with continuous boundary data that does not satisfy the compatibility condition into another Dirichlet problem that does satisfy the condition.

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u = f & \text{in } \{a\} \times [c,d] \\ u = g & \text{in } \{b\} \times [c,d] \\ u = h & \text{in } [a,b] \times \{d\} \\ u = k & \text{in } [a,b] \times \{c\} \end{cases} \qquad \begin{cases} \Delta \bar{u} = 0 & \text{in } R \\ \bar{u} = \bar{f} & \text{in } \{a\} \times [c,d] \\ \bar{u} = \bar{g} & \text{in } \{b\} \times [c,d] \\ \bar{u} = \bar{h} & \text{in } [a,b] \times \{d\} \\ \bar{u} = \bar{k} & \text{in } [a,b] \times \{c\} \end{cases}$$

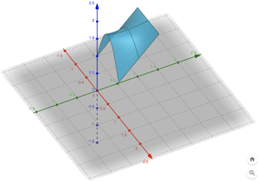
$$\bar{u} = u - P, \quad \bar{f} = f - P, \quad \bar{g} = g - P, \quad \bar{h} = h - P, \quad \bar{k} = k - P$$

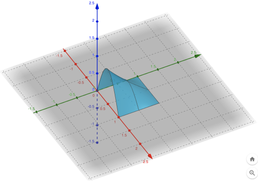
$$P(x,y) := a_o + a_1 x + a_2 y + a_3 xy$$

Note that $\bar{u}$ is still harmonic since P is harmonic, we can choose coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$ to ensure that the compatibility condition is fulfilled.

Laplace's Equation with Dirichlet boundary condition
Example 1

$$\begin{cases} \Delta u = 0 & \text{in } [0,1]\times[0,1] \\ u(x,0) = 1+\sin(\pi x) & \text{for } 0 \le x \le 1, \\ u(x,1) = 2 & \text{for } 0 \le x \le 1 \\ u(0,y) = 1+y & \text{for } 0 \le y \le 1 \\ u(1,y) = 1+y & \text{for } 0 \le y \le 1 \end{cases}$$





$P(x,y) = 1+y$

$\bar{u}(x,y) = u(x,y) - (1+y)$

Our construction implies that $\bar{u}$ is a harmonic function satisfying the compatability condition.

Start with the homogeneous direction:

$X_n(x) = \sin(n\pi x) \quad \lambda_n = n^2 \quad n = 1, 2, 3, \dots$

Laplace's Equation with Dirichlet boundary condition

Example 1

$$\begin{cases} \Delta u = 0 & \text{in } [0,1]\times[0,1] \\ u(x,0) = 1+\sin(\pi x) & \text{for } 0 \le x \le 1, \\ u(x,1) = 2 & \text{for } 0 \le x \le 1 \\ u(0,y) = 1+y & \text{for } 0 \le y \le 1 \\ u(1,y) = 1+y & \text{for } 0 \le y \le 1 \end{cases}$$

$$\begin{cases} \Delta \bar{u} = 0 \\ \bar{u}(x,0) = 1+\sin(\pi x) - (1+y)|_{y=0} = \sin(\pi x) \\ \bar{u}(x,1) = 2-(1+y)|_{y=1} = 0 \\ \bar{u}(0,y) = 1+y-(1+y) = 0 \\ \bar{u}(1,y) = 1+y-(1+y) = 0 \end{cases}$$

The other direction:

$$Y_n(y) = \alpha_n \sinh(n\pi y) + \beta_n \sinh(n\pi(y-1))$$

$$\bar{u}(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x)\left[A_n \sinh(n\pi y) + B_n \sinh(n\pi(y-1))\right]$$

$$\bar{u}(x,0) = \sum_{n=1}^{\infty} \sin(n\pi x)\cdot B_n \sinh(n\pi(-1)) = \sin(\pi x)$$

$$\begin{cases} B_1 = \dfrac{1}{\sinh(-\pi)} & n=1 \\ B_n = 0 & n \neq 1 \end{cases}$$

$\bar{u}(x,1) = 0$

$A_n = 0 \quad \forall n$

$$u(x,y) = \bar{u}(x,y) + (1+y) = \frac{1}{\sinh(\pi)}\sin(\pi x)\sinh(\pi(1-y)) + 1 + y$$

Compatibility condition
Laplace's Equation with Neumann boundary condition

Recall the necessary condition for the existence of a solution to the Neumann problem:

$$\begin{cases} \Delta u = \rho(x,y) & (x,y) \in D \\ \partial_v u(x,y) = g(x,y) & (x,y) \in \partial D \end{cases} \qquad \oint_{\partial D} g(x(s), y(s))\, ds = \int_D \rho(x,y)\, dx\, dy$$

heat flux through the boundary = heat generated in the domain

Laplace's equation in a rectangular domain with Neumann boundary conditions

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u_x = f & \text{on } \{a\} \times [c,d] \\ u_x = g & \text{on } \{b\} \times [c,d] \\ u_x = k & \text{on } [a,b] \times \{d\} \\ u_x = h & \text{on } [a,b] \times \{c\} \end{cases}$$

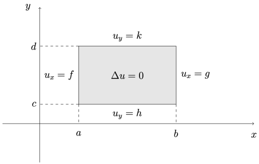


Figure 8.3: Neumann problem in a rectangular domain.

$$\int_c^d g - \int_c^d f + \int_a^b k - \int_a^b h = \int_c^d (g - f) + \int_a^b (k - h) = 0$$

Compatibility condition

Laplace's Equation with Neumann boundary condition

To solve the problem we need to split $u = u_1 + u_2$ in the sum of two problems as we did for the Dirichlet problem. Hence u_1, u_2 satisfy

$$\begin{cases} \Delta u_1 = 0 & \text{in } R \\ (u_1)_x = f & \text{on } \{a\} \times [c, d] \\ (u_1)_x = g & \text{on } \{b\} \times [c, d] \\ (u_1)_y = 0 & \text{on } [a, b] \times \{d\} \\ (u_1)_y = 0 & \text{on } [a, b] \times \{c\} \end{cases} \qquad \begin{cases} \Delta u_2 = 0 & \text{in } R \\ (u_2)_x = 0 & \text{on } \{a\} \times [c, d] \\ (u_2)_x = 0 & \text{on } \{b\} \times [c, d] \\ (u_2)_y = k & \text{on } [a, b] \times \{d\} \\ (u_2)_y = h & \text{on } [a, b] \times \{c\} \end{cases}$$

Note that, by splitting the problem, the existence condition for the Neumann problem might not be satisfied anymore for u_1 and u_2.

To overcome this problem, we use the trick of adding a harmonic polynomial $\alpha(x^2 - y^2)$ for some $\alpha \in \mathbb{R}$.

This yields the new harmonic function $v = u + \alpha(x^2 - y^2)$.

Compatibility condition

Laplace's Equation with Neumann boundary condition

If we now split $v = v_1 + v_2$ as we did above for u, then the problem for v_1 and v_2 are

$$\begin{cases} \Delta v_1 = 0 & \text{in } R \\ (v_1)_x = f + 2\alpha a & \text{on } \{a\} \times [c,d] \\ (v_1)_x = g + 2\alpha b & \text{on } \{b\} \times [c,d] \\ (v_1)_y = 0 & \text{on } [a,b] \times \{d\} \\ (v_1)_y = 0 & \text{on } [a,b] \times \{c\} \end{cases}$$

The compatibility condition for v_1

$$\int_c^d (g + 2\alpha b) - \int_c^d (f + 2\alpha a) = 0$$

$$\alpha = \frac{1}{2(b-a)(d-c)} \int_c^d (f - g)$$

$$\begin{cases} \Delta v_2 = 0 & \text{in } R \\ (v_2)_x = 0 & \text{on } \{a\} \times [c,d] \\ (v_2)_x = 0 & \text{on } \{b\} \times [c,d] \\ (v_2)_y = k - 2\alpha d & \text{on } [a,b] \times \{d\} \\ (v_2)_y = h - 2\alpha c & \text{on } [a,b] \times \{c\} \end{cases}$$

The compatibility condition for v_2

$$\int_a^b (k - 2\alpha d) - \int_a^b (h - 2\alpha c) = 0$$

$$\alpha = \frac{1}{2(b-a)(d-c)} \int_a^b (k - h)$$

Laplace's Equation with Neumann boundary condition

Example 2

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi] \\ u_x(0,y) = 0 & \text{on } 0\le y\le \pi \\ u_x(\pi,y) = \sin(y) & \text{on } 0\le y\le \pi \\ u_y(x,0) = 0 & \text{on } 0\le x\le \pi \\ u_y(x,\pi) = -\sin(x) & \text{on } 0\le x\le \pi \end{cases}$$

$$\begin{cases} \Delta v = 0 \\ v_x(0,y) = 0 \\ v_x(\pi,y) = \sin(y) + 2\alpha\pi \\ v_y(x,0) = 0 \\ v_y(x,\pi) = -\sin(x) - 2\alpha\pi \end{cases}$$

$$v(x,y) = u(x,y) + \alpha(x^2 - y^2)$$

$$\alpha = \frac{1}{2(b-a)\cdot(d-c)}\int_c^d f-g = \frac{1}{2\pi^2}\int_0^\pi -\sin(y)\,dy = -\frac{1}{\pi^2}$$

$$\alpha = \frac{1}{2(b-a)\cdot(d-c)}\int_a^b k-h = \frac{1}{2\pi^2}\int_0^\pi -\sin(x)\,dx = -\frac{1}{\pi^2}$$

These two results should coincide

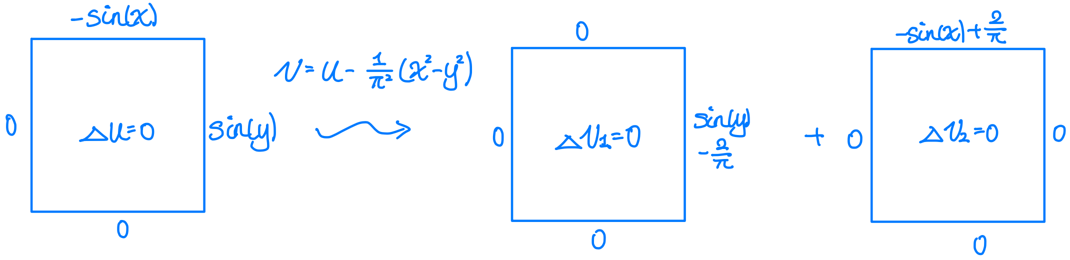
Laplace's Equation with Neumann boundary condition

Example 2

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi] \\ u_x(0,y) = 0 & \text{on } 0 \le y \le \pi \\ u_x(\pi,y) = \sin(y) & \text{on } 0 \le y \le \pi \\ u_y(x,0) = 0 & \text{on } 0 \le x \le \pi \\ u_y(x,\pi) = -\sin(x) & \text{on } 0 \le x \le \pi \end{cases}$$

$$\begin{cases} \Delta v_1 = 0 \\ (v_1)_x(0,y) = 0 \\ (v_1)_x(\pi,y) = \sin(y) - \frac{2}{\pi} \\ (v_1)_y(x,0) = 0 \\ (v_1)_y(x,\pi) = 0 \end{cases}
\qquad
\begin{cases} \Delta v_2 = 0 \\ (v_2)_x(0,y) = 0 \\ (v_2)_x(\pi,y) = 0 \\ (v_2)_y(x,0) = 0 \\ (v_2)_y(x,\pi) = -\sin(x) + \frac{2}{\pi} \end{cases}$$

$$v = u - \frac{1}{\pi^2}(x^2 - y^2)$$



Outline

Laplace's equation on circular domains

Let B_a be a disk of radius a around the origin, the Dirichlet problem is:

$$\begin{cases} \Delta u = 0 & (x,y) \in B_a \\ u(x,y) = g(x,y) & (x,y) \in \partial B_a \end{cases}$$

It is convenient to solve the equation in polar coordinates

$$w(r,\theta) = u(x(r,\theta), y(r,\theta))$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\Delta w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}$$

$$\begin{cases} w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} = 0 & 0 \le r < a,\ 0 \le \theta \le 2\pi \\ w(a,\theta) = h(\theta) = g(x(a,\theta), y(a,\theta)) & r = a, \qquad 0 \le \theta \le 2\pi \end{cases}$$

Laplace's equation on circular domains

$$w(r,\theta) = R(r)\Theta(\theta)$$

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

Solve ODE for $\Theta(\theta)$ first

$$\begin{cases} \Theta''(\theta) & = -\lambda\Theta(\theta) \\ \Theta(0) & = \Theta(2\pi) \\ \Theta'(0) & = \Theta'(2\pi) \end{cases}$$

$$\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \;\text{ for } \lambda_n = n^2, \;\; n = 0, 1, 2, ...$$

Laplace's equation on circular domains

Then solve the ODE for $R(r)$ together with the eigenvalue $\lambda_n = n^2$:

$$r^2 R_n'' + rR_n' - n^2 R_n = 0$$
$$R_n(r) = C_n r^n + D_n r^{-n}, \;\; n = 1, 2, 3, ...$$

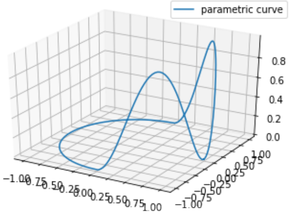
$$R_n(r) = \begin{cases} C_0 + D_0 \ln(r) & n = 0 \\ C_n r^n + D_n r^{-n} & n \neq 0 \end{cases}$$

However, the fucntion r^{-n} and $ln(r)$ are singular at 0 inside the domain D, so we discard them.
Thus the general solution is given by:

$$w(r, \theta) = C_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

Laplace's equation on circular domains
Example 3

$$\begin{cases} \Delta u(r,\theta) = 0 & 0 < r < R,\ 0 < \theta < 2\pi \\ u(R,\theta) = \sin^2(2\theta) & 0 \le \theta < \frac{\pi}{2}, \\ u(R,\theta) = 0 & \frac{\pi}{2} \le \theta < \frac{\pi}{2}, \\ u(R,\theta) = \sin^2(2\theta) & \frac{3\pi}{2} \le \theta < 2\pi, \end{cases}$$



Evaluate $u(0,0)$ without solving the PDE.

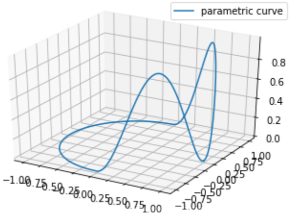
The mean value theorem:

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \sin^2(2\theta)\,d\theta = \frac{1}{4}$$

Laplace’s equation on circular domains

Example 3

$$\begin{cases} \Delta u(r,\theta) = 0 & 0 < r < R,\ 0 < \theta < 2\pi \\ u(R,\theta) = \sin^2(2\theta) & 0 \le \theta < \frac{\pi}{2}, \\ u(R,\theta) = 0 & \frac{\pi}{2} \le \theta < \frac{\pi}{2}, \\ u(R,\theta) = \sin^2(2\theta) & \frac{3\pi}{2} \le \theta < 2\pi, \end{cases}$$



Show that the inequality $0 < u(r,\theta) < 1$ holds at each point (r,θ) in the disk.

The weak maximum principle:

$u(r,\theta) \le \max_{\theta\in[0,2\pi]} u(R,\theta) = 1$ for all $r < R$, and the equality holds if and only if u is constant. The solution is not a constant function, and therefore $u < 1$ in D.

The inequality $u > 0$ is obtained from the maximum principles applied to $-u$.

Outline

Tips for Serie 12

1. Separation of variables for elliptic equations
   - Compatibility condition for Laplace's equation with Neumann boundary condition.
   - Use the correct direction to integrate.
2. Separation of variables
   - Solve the homogeneous direction at first.
   - Then proceed with the other direction.
   - Case studies according to the eigenvalues.
3. Neumann problem
   - Consult example 2.
4. Laplace operator and rotations
   - Express x and y in terms of s and t.
   - Calculate $\Delta v(s, t)$ using the chain rule.

Self-promotion:

Teaching Assistant for ***Introduction to Machine Learning*** from D-INFK next semester

Instructor: Prof. Dr. Andreas Krause and Prof. Dr. Fan Yang

The course introduces the foundations of learning and making predictions from data.

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein