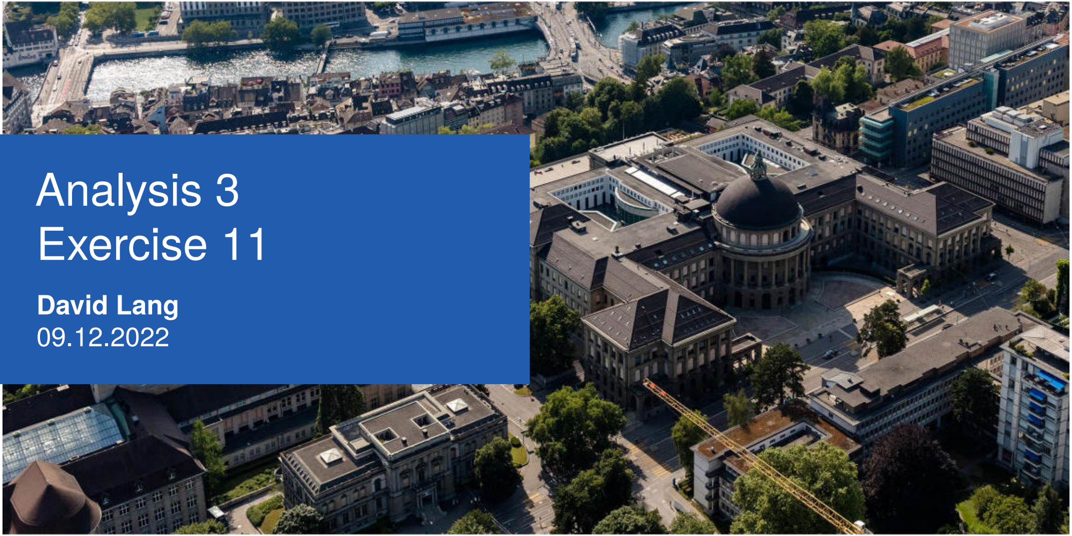

Analysis 3
Exercise 11

David Lang
09.12.2022

Outline

1. Serie 10 Review
2. Course Overview
3. Maximum Principle for parabolic equations
4. Separation of Variables for elliptic equations
5. Tips for Serie 11

Outline

Serie 10 Review

1. Unique solution
   - Maximum principles are based on the observation that, if a C^2 function u attains its maximum over an open set D at a point $\vec{x_0} \in D$, then $Du(\vec{x_0}) = 0$, and $D^2u(\vec{x_0})$ is negative semidefinite.
2. The mean-value principle
3. Maximum principle

$$\int_0^{2\pi} cos^2(\theta)\, d\theta = \pi, \quad \int_0^{2\pi} cos(\theta)\, d\theta = 0$$

4. Multiple choice

$$\int_D \rho = \int_0^R r \int_0^{2\pi} r^\alpha \sin^2(\theta)\, d\theta\, dr = \pi \frac{R^{\alpha+2}}{\alpha+2}$$

$$\int_{\partial D} g = \int_0^{2\pi} R\Big(C\cos^2(\theta) + R^{2021}\sin(\theta)\Big)\, d\theta = RC\pi$$

5. Weak maximum principle

Outline

1. Serie 10 Review
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3. Maximum Principle for parabolic equations
4. Separation of Variables for elliptic equations
5. Tips for Serie 11

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws

- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - **Heat equation**
    - **Maximum principle**
    - Separation of variables
  - Elliptic PDEs
    - **Laplace equation**
    - Maximum principle
    - **Separation of variables**

Outline

Maximum Principle for heat Equation

Consider the heat equation for $u = u(t, \vec{x}), t > 0, \vec{x} \in D$, namely

$$u_t = k\Delta u$$

Define the parabolic boundary as

$$\partial_P Q_T := \{\{0\} \times D\} \cup \{[0,T] \times \partial D\}$$

Let u solve the homogeneous heat equation, and $D \in \mathbb{R}^2$ bounded, then u achieves its maximum (and minimum) on $\partial_P Q_T$.

Maximum Principle for heat Equation

Example 1

Let $D \in \mathbb{R}^2$ be a bounded domain, $T > 0$, and set $Q_T = D \times [0, T]$.

Let $u : Q_T \to \mathbb{R}$ be a classical solution to the PDE

$$\begin{cases} u_t = u_{xx} + x^2 u_{yy} + u_y & \text{for } (x, y, t) \in Q_T \\ u(x, y, t) = g(x, y, t) & \text{on } \partial D \times [0, T] \\ u(x, y, 0) = f(x, y) & \text{on } D \end{cases}$$

namely, u is twice differentiable with respect to (x, y) in Q_T, once differentiable with respect to t in Q_T, and continuous in $\bar{Q_T}$.

Prove that u attains its minimum on the parabolic boundary

$$\partial_P Q_T := (\{0\} \times D) \cup ([0, T] \times \partial D)$$

Hint: consider $v(x, y, t) = u(x, y, t) + \epsilon t$ with $\epsilon > 0$, and prove that v can attain a minimum only on $\partial_p Q_T$.

Maximum Principle for heat Equation
Example 1

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namely, u is twice differentiable with respect to (x, y) in Q_T, once differentiable with respect to t in Q_T, and continuous in $\bar{Q_T}$.

Prove that, given f and g, there is at most one classical solution.

Maximum Principle for heat Equation
Example 1

Outline

Separation of Variables for elliptic equations

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u = 0 & \text{in } [a,b] \times \{c,d\} \\ u = f & \text{in } \{a\} \times [c,d] \\ u = g & \text{in } \{b\} \times [c,d] \end{cases}$$

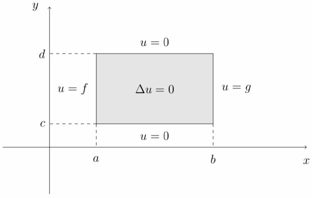

Figure 8.1: Laplace equation in a rectangular domain.

$$u(x,y) = X(x)Y(y)$$

$$X''(x)Y(y) + Y''(y)X(x) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

In this case Y is the homogeneous direction and X is the inhomogeneous direction.

Separation of Variables for elliptic equations

$$\begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(c) = Y(d) = 0 \end{cases}$$

$$Y_n(y) = \sin\left(\frac{n\pi(y-c)}{d-c}\right) \quad \lambda_n = \left(\frac{n\pi}{d-c}\right)^2 \quad n = 1, 2, 3, ...$$

$$X_n''(x) = \lambda_n X_n(x)$$

$$X_n(x) = \alpha_n \sinh\left(\frac{n\pi(x-a)}{d-c}\right) + \beta_n \sinh\left(\frac{n\pi(x-b)}{d-c}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} \left[A_n \sinh\left(\frac{n\pi(x-a)}{d-c}\right) + B_n \sinh\left(\frac{n\pi(x-b)}{d-c}\right)\right] \sin\left(\frac{n\pi(y-c)}{d-c}\right)$$

General Procedure for Laplace's equation on rectangular domains

1. Check the compatibility condition for the existence of a solution.
   - (Details next week)
2. $u = 0$ on two opposite sides of the rectangle?
   - Yes –> No modification
   - No –> Split into two sub-problems and check the compatibility condition again
3. Solve the homogeneous direction
4. Solve the non-homogeneous direction
5. Combine the solutions

Laplace's equation on rectangular domains

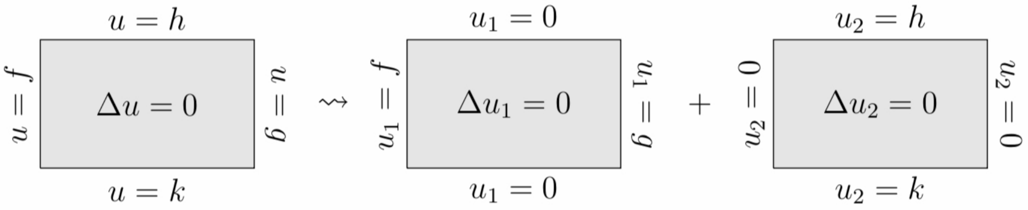

Figure 8.2: Splitting of the Laplace equation in a rectangular domain.

Laplace's equation in rectangular domains

Eigenfunctions in the homogeneous direction:

Dirichlet:

$$u_1 \ (y - direction) : Y_n(y) = \sin\left(\frac{n\pi(y-c)}{d-c}\right) \quad \lambda_n = \left(\frac{n\pi}{d-c}\right)^2 \ \ n = 1, 2, 3, ...$$

$$u_2 \ (x - direction) : X_n(x) = \sin\left(\frac{n\pi(x-a)}{b-a}\right) \quad \lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \ \ n = 1, 2, 3, ...$$

Neumann:

$$u_1 \ (y - direction) : Y_n(y) = \cos\left(\frac{n\pi(y-c)}{d-c}\right) \quad \lambda_n = \left(\frac{n\pi}{d-c}\right)^2 \ \ n = 0, 1, 2, ...$$

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Laplace's equation on rectangular domains
Continued

In the other direction:

Dirichlet:

$$u_1\ (x - direction) : X_n(x) = \alpha_n \sinh\left(\frac{n\pi(x-a)}{d-c}\right) + \beta_n \sinh\left(\frac{n\pi(x-b)}{d-c}\right)$$

$$u_2\ (y - direction) : Y_n(y) = \alpha_n \sinh\left(\frac{n\pi(y-c)}{b-a}\right) + \beta_n \sinh\left(\frac{n\pi(y-d)}{b-a}\right)$$

Neumann:

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Laplace Equation with Dirichlet boundary condition
Example 2

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\ u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\ u(x,\pi) = \sin(x) + \frac{1}{2}\sin(2x) & \text{for } 0 \le x \le \pi, \\ u(0,y) = u(\pi,y) = 0 & \text{for } 0 \le y \le \pi \end{cases}$$

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Laplace Equation with Dirichlet boundary condition

Example 3

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\ u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\ u(x,\pi) = 0 & \text{for } 0 \le x \le \pi, \\ u(0,y) = \sin(y) + \frac{1}{2}\sin(2y) & \text{for } 0 \le y \le \pi, \\ u(\pi,y) = 0 & \text{for } 0 \le y \le \pi \end{cases}$$

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Laplace Equation with Dirichlet boundary condition
Example 3

$$
\begin{cases}
\Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\
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u(0,y) = \sin(y) + \frac{1}{2}\sin(2y) & \text{for } 0 \le y \le \pi, \\
u(\pi,y) = 0 & \text{for } 0 \le y \le \pi
\end{cases}
$$

Outline

Tips for Serie 11

1. Separation of variables for elliptic equations
   - (a) Solve the homogeneous direction first
   - (b) Modify (add or subtract) to get Laplace's Equation
2. Heat Equation
   - $\sin(\pi x) \geq x(1-x)$ in the interval $[0, 1]$
   - How does the initial value effect the evolution?
3. Uniqueness of solutions
   - Similar to 10.1

ETH zürich

Self-promotion:

Teaching Assistant for ***Introduction to Machine Learning*** from D-INFK next semester

Instructor: Prof. Dr. Andreas Krause and Prof. Dr. Fan Yang

The course introduces the foundations of learning and making predictions from data.

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein