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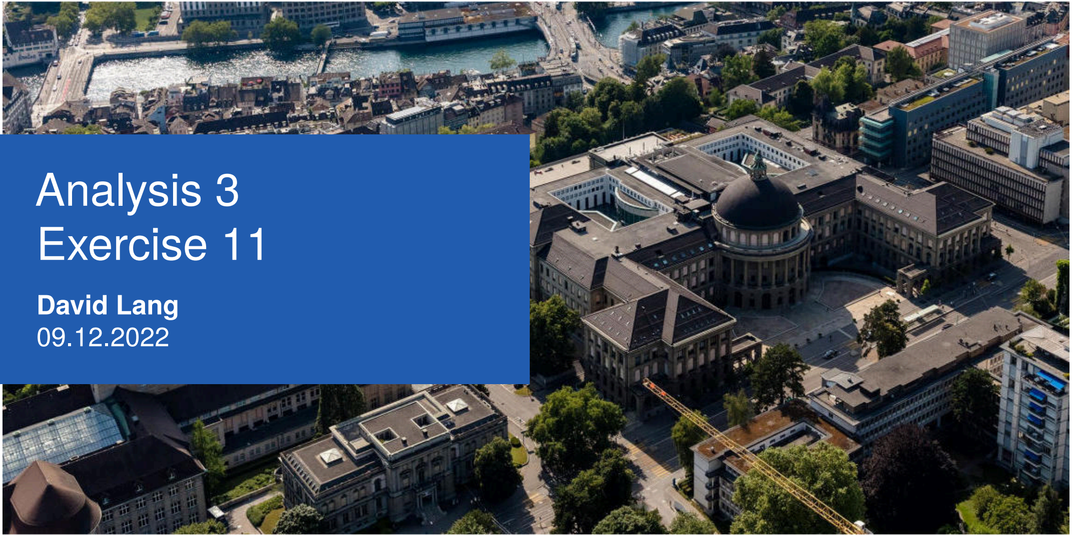

Analysis 3
Exercise 11

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09.12.2022

Outline

1. Serie 10 Review

2. Course Overview

3. Maximum Principle for parabolic equations

4. Separation of Variables for elliptic equations

5. Tips for Serie 11

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Serie 10 Review

1. Unique solution
   - Maximum principles are based on the observation that, if a C^2 function u attains its maximum over an open set D at a point $\vec{x_0} \in D$, then $Du(\vec{x_0}) = 0$, and $D^2u(\vec{x_0})$ is negative semidefinite.
2. The mean-value principle
3. Maximum principle

$$\int_0^{2\pi} cos^2(\theta)\,d\theta = \pi, \quad \int_0^{2\pi} cos(\theta)\,d\theta = 0$$

4. Multiple choice

$$\int_D \rho = \int_0^R r \int_0^{2\pi} r^\alpha \sin^2(\theta)\,d\theta\,dr = \pi \frac{R^{\alpha+2}}{\alpha+2}$$

$$\int_{\partial D} g = \int_0^{2\pi} R\Big(C\cos^2(\theta) + R^{2021}\sin(\theta)\Big)\,d\theta = RC\pi$$

5. Weak maximum principle

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Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - **Heat equation**
    - **Maximum principle**
    - Separation of variables
  - Elliptic PDEs
    - **Laplace equation**
    - Maximum principle
    - **Separation of variables**

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Maximum Principle for heat Equation

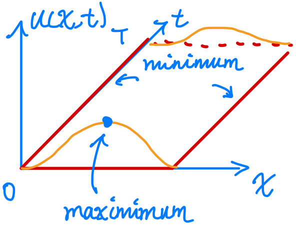
Consider the heat equation for $u = u(t, \vec{x}), t > 0, \vec{x} \in D$, namely

$$u_t = k\Delta u$$

Define the parabolic boundary as

$$\partial_P Q_T := \{\{0\} \times D\} \cup \{[0, T] \times \partial D\}$$

Let u solve the homogeneous heat equation, and $D \in \mathbb{R}^2$ bounded, then u achieves its maximum (and minimum) on $\partial_P Q_T$.



Maximum Principle for heat Equation
Example 1

Let $D \in \mathbb{R}^2$ be a bounded domain, $T > 0$, and set $Q_T = D \times [0, T]$.

Let $u : Q_T \to \mathbb{R}$ be a classical solution to the PDE

$$\begin{cases} u_t = u_{xx} + x^2 u_{yy} + u_y & \text{for } (x, y, t) \in Q_T \\ u(x, y, t) = g(x, y, t) & \text{on } \partial D \times [0, T] \\ u(x, y, 0) = f(x, y) & \text{on } D \end{cases}$$

namely, u is twice differentiable with respect to (x, y) in Q_T, once differentiable with respect to t in Q_T, and continuous in $\bar{Q_T}$.

Prove that u attains its minimum on the parabolic boundary

$$\partial_P Q_T := (\{0\} \times D) \cup ([0, T] \times \partial D)$$

Hint: consider $v(x, y, t) = u(x, y, t) + \epsilon t$ with $\epsilon > 0$, and prove that v can attain a minimum only on $\partial_p Q_T$.

Maximum Principle for heat Equation

Example 1

Want: $\min_{\partial_p Q_T} u \leq \min_{\overline{Q_T}} u$

Hint: $\min_{\partial_p Q_T} v \leq \min_{\overline{Q_T}} v \qquad v(x,y,t) = u(x,y,t) + \varepsilon t$

$$\min_{\partial_p Q_T} u \leq \min_{\partial_p Q_T} (u + \varepsilon t) \leq \min_{\overline{Q_T}} (u + \varepsilon t) \leq \left(\min_{\overline{Q_T}} u\right) + \varepsilon T$$

$\varepsilon \to 0$: $\min_{\partial_p Q_T} u \leq \min_{\overline{Q_T}} u$

Goal: $\min_{\partial_p Q_T} v \leq \min_{\overline{Q_T}} v$

Suppose v attains its minimum at $(x_0, y_0, t_0) \in \overline{Q_T} \setminus \partial_p Q_T$

Two cases: $t_0 < T$ or $t_0 = T$

Maximum Principle for heat Equation

Example 1

assumption: (x_0, y_0, t_0) its a minimum

$t_0 < T$ $\quad \nabla v(x_0,y_0,t_0)=0 \quad v_t(x_0,y_0,t_0)=0 \quad v_{xx}(x_0,y_0,t_0)\geq 0 \quad v_{yy}(x_0,y_0,t_0)\geq 0$

$$v_t = u_t + \varepsilon = u_{xx} + x^2 u_{yy} + u_y + \varepsilon$$

$$= v_{xx} + x^2 v_{yy} + v_y + \varepsilon$$

$$0 = v_t \geq \varepsilon > 0 \quad \lightning$$

$t_0 = T$ $\quad \nabla v(x_0,y_0,t_0)=0 \quad v_t(x_0,y_0,t_0)\leq 0 \quad v_{xx}(x_0,y_0,t_0)\geq 0 \quad v_{yy}(x_0,y_0,t_0)\geq 0$

$$v_t = u_t + \varepsilon = u_{xx} + x^2 u_{yy} + u_y + \varepsilon$$

$$= v_{xx} + x^2 v_{yy} + v_y + \varepsilon$$

$$0 \geq v_t \geq \varepsilon > 0 \quad \lightning$$

$\Rightarrow$ u attains its minimum on the parabolic boundary.

Maximum Principle for heat Equation
Example 1

Let $D \in \mathbb{R}^2$ be a bounded domain, and set $Q_T = D \times [0, T]$.

Let $u : Q_T \to \mathbb{R}$ be a classical solution to the PDE

$$\begin{cases} u_t = u_{xx} + x^2 u_{yy} + u_y & \text{for } (x, y, t) \in Q_T \\ u(x, y, t) = g(x, y, t) & \text{on } \partial D \times [0, T] \\ u(x, y, 0) = f(x, y) & \text{on } D \end{cases}$$

namely, u is twice differentiable with respect to (x, y) in Q_T, once differentiable with respect to t in Q_T, and continuous in $\bar{Q_T}$.

Prove that, given f and g, there is at most one classical solution.

Maximum Principle for heat Equation

Example 1

Let u_1, u_2 be two solutions.

$w := u_1 - u_2$

$$\begin{cases} w_t = w_{xx} + x^2 w_{yy} + w_y & \text{in } Q_T \\ w(x,y,t) = 0 & \text{on } \partial D \times [0,T] \\ w(x,y,0) = 0 & \text{on } D \end{cases}$$

$v := u_2 - u_1$

$$\begin{cases} v_t = v_{xx} + x^2 v_{yy} + v_y & \text{in } Q_T \\ v(x,y,t) = 0 & \text{on } \partial D \times [0,T] \\ v(x,y,0) = 0 & \text{on } D \end{cases}$$

w & v attains its minimum on the parabolic boundary

$w \geq 0$ in Q_T

$u_1 \geq u_2$

$v \geq 0$ in Q_T

$u_2 \geq u_1$

$$\Rightarrow u_1 \equiv u_2$$

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Separation of Variables for elliptic equations

$$\begin{cases} \Delta u = 0 & \text{in } R \\ u = 0 & \text{in } [a,b] \times \{c,d\} \\ u = f & \text{in } \{a\} \times [c,d] \\ u = g & \text{in } \{b\} \times [c,d] \end{cases}$$

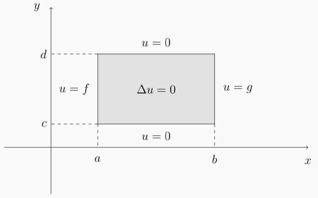


Figure 8.1: Laplace equation in a rectangular domain.

$$u(x,y) = X(x)Y(y)$$

$$X''(x)Y(y) + Y''(y)X(x) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

In this case Y is the homogeneous direction and X is the inhomogeneous direction.

Separation of Variables for elliptic equations

$$\begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(c) = Y(d) = 0 \end{cases}$$

$$Y_n(y) = \sin\left(\frac{n\pi(y-c)}{d-c}\right) \quad \lambda_n = \left(\frac{n\pi}{d-c}\right)^2 \quad n = 1, 2, 3, ...$$

$$X_n''(x) = \lambda_n X_n(x)$$

$$X_n(x) = \alpha_n \sinh\left(\frac{n\pi(x-a)}{d-c}\right) + \beta_n \sinh\left(\frac{n\pi(x-b)}{d-c}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty}\left[A_n \sinh\left(\frac{n\pi(x-a)}{d-c}\right) + B_n \sinh\left(\frac{n\pi(x-b)}{d-c}\right)\right]\sin\left(\frac{n\pi(y-c)}{d-c}\right)$$

General Procedure for Laplace's equation on rectangular domains

1. Check the compatibility condition for the existence of a solution.
   - (Details next week)
2. $u = 0$ on two opposite sides of the rectangle?
   - Yes –> No modification
   - No –> Split into two sub-problems and check the compatibility condition again
3. Solve the homogeneous direction
4. Solve the non-homogeneous direction
5. Combine the solutions

Laplace's equation on rectangular domains

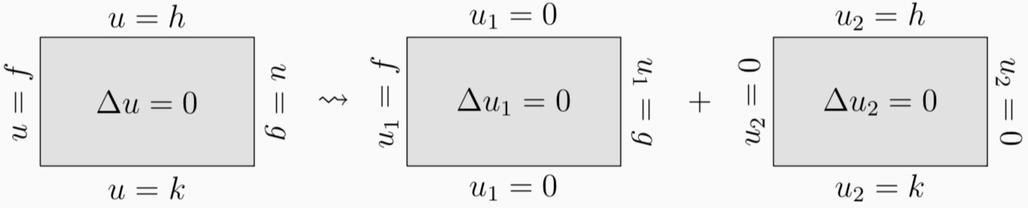


Figure 8.2: Splitting of the Laplace equation in a rectangular domain.

Laplace's equation in rectangular domains

Eigenfunctions in the homogeneous direction:

Dirichlet:

$$u_1\ (y - direction) : Y_n(y) = \sin\left(\frac{n\pi(y-c)}{d-c}\right) \quad \lambda_n = \left(\frac{n\pi}{d-c}\right)^2 \quad n = 1, 2, 3, ...$$

$$u_2\ (x - direction) : X_n(x) = \sin\left(\frac{n\pi(x-a)}{b-a}\right) \quad \lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \quad n = 1, 2, 3, ...$$

Neumann:

$$u_1\ (y - direction) : Y_n(y) = \cos\left(\frac{n\pi(y-c)}{d-c}\right) \quad \lambda_n = \left(\frac{n\pi}{d-c}\right)^2 \quad n = 0, 1, 2, ...$$

$$u_2\ (x - direction) : X_n(x) = \cos\left(\frac{n\pi(x-a)}{b-a}\right) \quad \lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \quad n = 0, 1, 2, ...$$

Laplace's equation on rectangular domains
Continued

In the other direction:

Dirichlet:

$$u_1\ (x - direction) : X_n(x) = \alpha_n \sinh\left(\frac{n\pi(x-a)}{d-c}\right) + \beta_n \sinh\left(\frac{n\pi(x-b)}{d-c}\right)$$

$$u_2\ (y - direction) : Y_n(y) = \alpha_n \sinh\left(\frac{n\pi(y-c)}{b-a}\right) + \beta_n \sinh\left(\frac{n\pi(y-d)}{b-a}\right)$$

Neumann:

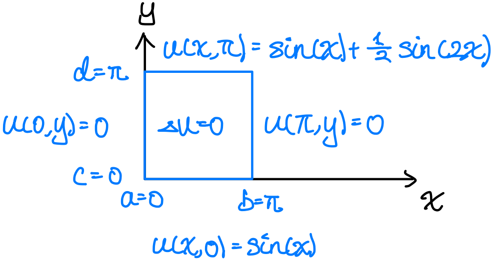
$$u_1\ (x - direction) : X_n(x) = \alpha_n \cosh\left(\frac{n\pi(x-a)}{d-c}\right) + \beta_n \cosh\left(\frac{n\pi(x-b)}{d-c}\right)$$

$$u_2\ (y - direction) : Y_n(y) = \alpha_n \cosh\left(\frac{n\pi(y-c)}{b-a}\right) + \beta_n \cosh\left(\frac{n\pi(y-d)}{b-a}\right)$$

Laplace Equation with Dirichlet boundary condition

Example 2

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\ u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\ u(x,\pi) = \sin(x) + \frac{1}{2}\sin(2x) & \text{for } 0 \le x \le \pi, \\ u(0,y) = u(\pi,y) = 0 & \text{for } 0 \le y \le \pi \end{cases}$$



$u(x,y) = X(x)Y(y)$

$X_n(x) = \sin(nx) \qquad \lambda_n = n^2 \qquad n = 1,2,3,\dots$

$Y_n(y) = \alpha_n \sinh(ny) + \beta_n \sinh(n(y-\pi))$

$u(x,y) = \sum_{n=1}^{\infty} \sin(nx)\left[A_n \sinh(ny) + B_n \sinh(n(y-\pi))\right]$

Laplace Equation with Dirichlet boundary condition

Example 2

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\ u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\ u(x,\pi) = \sin(x) + \frac{1}{2}\sin(2x) & \text{for } 0 \le x \le \pi, \\ u(0,y) = u(\pi,y) = 0 & \text{for } 0 \le y \le \pi \end{cases}$$

$$u(x, y=0) = \sum_{n=1}^{\infty} \sin(nx) B_n \sinh(-n\pi) = \sin(x)$$

$$\begin{cases} B_1 = \dfrac{1}{\sinh(-\pi)} & n=1 \\ B_n = 0 & n \neq 1 \end{cases}$$

$$u(x, y=\pi) = \sum_{n=1}^{\infty} \sin(nx) A_n \sinh(n\pi) = \sin(x) + \frac{1}{2}\sin(2x)$$

$$\begin{cases} A_1 = \dfrac{1}{\sinh(\pi)} & n=1 \\ A_2 = \dfrac{1}{2\sinh(2\pi)} & n=2 \\ A_n = 0 & n \neq 1,2 \end{cases}$$

Laplace Equation with Dirichlet boundary condition

Example 2

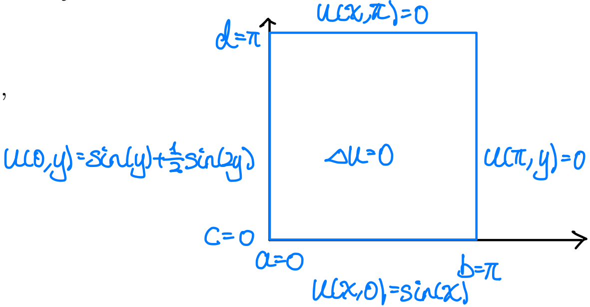
$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\ u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\ u(x,\pi) = \sin(x) + \frac{1}{2}\sin(2x) & \text{for } 0 \le x \le \pi, \\ u(0,y) = u(\pi,y) = 0 & \text{for } 0 \le y \le \pi \end{cases}$$

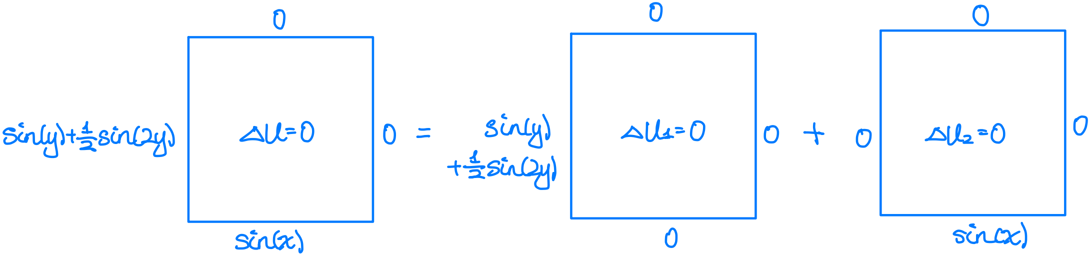
$$u(x,y) = \frac{1}{\sinh(\pi)}\sin(x)\sinh(y) - \frac{1}{\sinh(\pi)}\sin(x)\sinh(y-\pi) + \frac{1}{2\sinh(2\pi)}\sin(2x)\sinh(2y)$$

Laplace Equation with Dirichlet boundary condition

Example 3

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\ u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\ u(x,\pi) = 0 & \text{for } 0 \le x \le \pi, \\ u(0,y) = \sin(y) + \frac{1}{2}\sin(2y) & \text{for } 0 \le y \le \pi, \\ u(\pi,y) = 0 & \text{for } 0 \le y \le \pi \end{cases}$$

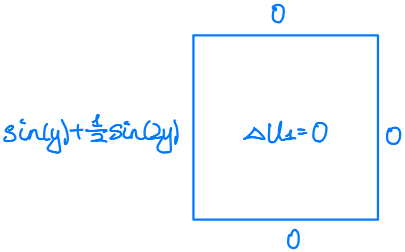




Laplace Equation with Dirichlet boundary condition

Example 3

$$
\begin{cases}
\Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\
u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\
u(x,\pi) = 0 & \text{for } 0 \le x \le \pi, \\
u(0,y) = \sin(y) + \frac{1}{2}\sin(2y) & \text{for } 0 \le y \le \pi, \\
u(\pi,y) = 0 & \text{for } 0 \le y \le \pi
\end{cases}
$$



$Y_n(y) = \sin(ny) \qquad \lambda_n = n^2 \qquad n = 1,2,3,\dots$

$X_n(x) = \alpha_n \sinh(nx) + \beta_n \sinh(n(x-\pi))$

$U_1(x,y) = \sum_{n=1}^{\infty} \sin(ny)\left[A_n \sinh(nx) + B_n \sinh(n(x-\pi))\right]$

$U_1(0,y) = \sum_{n=1}^{\infty} \sin(ny) B_n \sinh(-n\pi) = \sin(y) + \frac{1}{2}\sin(2y)$

$$
\begin{cases}
B_1 = \frac{1}{\sinh(-\pi)} & n = 1 \\
B_2 = \frac{1}{2\sinh(-2\pi)} & n = 2 \\
B_n = 0 & n \neq 1,2
\end{cases}
$$

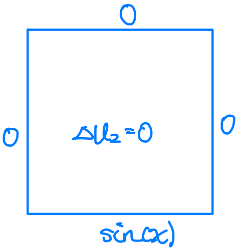
$U_1(\pi,y) = \sum_{n=1}^{\infty} \sin(ny) A_n \sinh(n\pi) = 0 \qquad A_n = 0 \quad \forall n$

$U_1(x,y) = -\frac{1}{\sinh(\pi)} \sin(y)\sinh(x-\pi) - \frac{1}{2\sinh(2\pi)} \sin(2y)\sinh(2(x-\pi))$

Laplace Equation with Dirichlet boundary condition

Example 3

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi], \\ u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\ u(x,\pi) = 0 & \text{for } 0 \le x \le \pi, \\ u(0,y) = \sin(y) + \frac{1}{2}\sin(2y) & \text{for } 0 \le y \le \pi, \\ u(\pi,y) = 0 & \text{for } 0 \le y \le \pi \end{cases}$$



$X_n(x) = \sin(nx) \qquad \lambda_n = n^2 \qquad n = 1, 2, 3, \ldots$

$Y_n(y) = \alpha_n \sinh(ny) + \beta_n \sinh(n(y-\pi))$

$u_2(x,y) = \sum_{n=1}^{\infty} \sin(nx)\left[A_n \sinh(ny) + B_n \sinh(n(y-\pi))\right]$

$u_2(x,0) = \sum_{n=1}^{\infty} \sin(nx) B_n \sinh(-n\pi) = \sin(x)$

$\begin{cases} B_1 = \frac{1}{\sinh(-\pi)} & n = 1 \\ B_n = 0 & n \neq 1 \end{cases}$

$u_2(x,\pi) = \sum_{n=1}^{\infty} \sin(nx) A_n \sinh(n\pi) = 0$

$A_n = 0 \quad \forall n$

$u_2(x,y) = -\frac{1}{\sinh(\pi)} \sin(x) \sinh(y-\pi)$

Laplace Equation with Dirichlet boundary condition

Example 3

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi] \times [0,\pi], \\ u(x,0) = \sin(x) & \text{for } 0 \le x \le \pi, \\ u(x,\pi) = 0 & \text{for } 0 \le x \le \pi, \\ u(0,y) = \sin(y) + \frac{1}{2}\sin(2y) & \text{for } 0 \le y \le \pi, \\ u(\pi,y) = 0 & \text{for } 0 \le y \le \pi \end{cases}$$

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

$$= -\frac{1}{\sinh(\pi)}\sin(x)\sinh(y-\pi) - \frac{1}{\sinh(\pi)}\sin(y)\sinh(x-\pi)$$

$$-\frac{1}{2\sinh(2\pi)}\sin(2y)\sinh(2(x-\pi))$$

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Tips for Serie 11

1. Separation of variables for elliptic equations
   - (a) Solve the homogeneous direction first
   - (b) Modify (add or subtract) to get Laplace's Equation
2. Heat Equation
   - $\sin(\pi x) \geq x(1-x)$ in the interval $[0, 1]$
   - How does the initial value effect the evolution?
3. Uniqueness of solutions
   - Similar to 10.1

Self-promotion:

Teaching Assistant for ***Introduction to Machine Learning*** from D-INFK next semester

Instructor: Prof. Dr. Andreas Krause and Prof. Dr. Fan Yang

The course introduces the foundations of learning and making predictions from data.

References:

1. Lecture notes on the course website.
2. "An Introduction to Partial Differential Equations" by Yehuda Pinchover and Jacob Rubinstein