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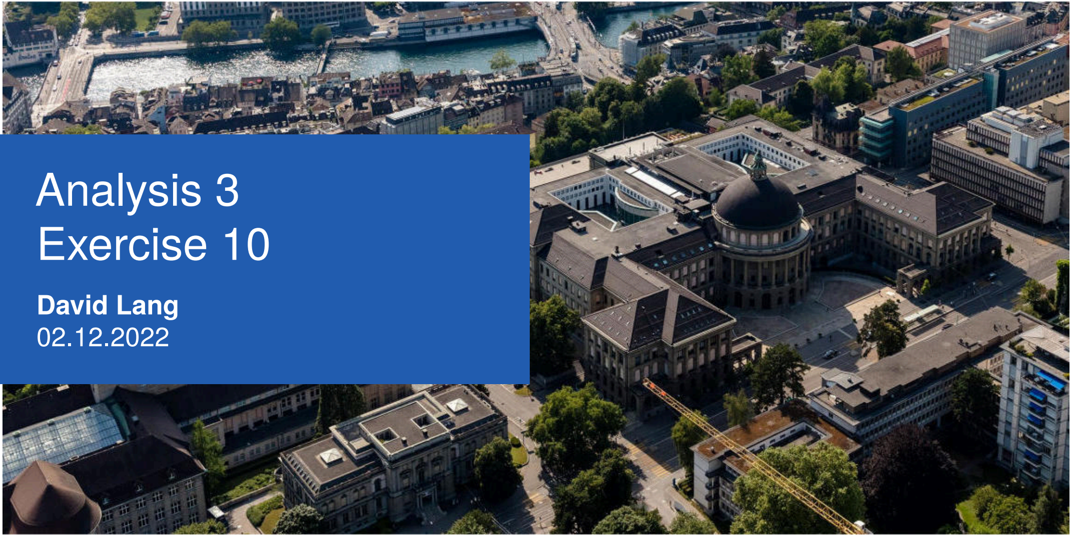

Analysis 3
Exercise 10

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Outline

1. Serie 9 Review
2. Course Overview
3. From Gauss' Law to Poisson's and Laplace's Equations
4. The Intuition behind Laplace's Equation
5. Poisson's and Laplace's Equation and boundary conditions
6. Maximum Principles
7. Tips for Serie 10

Outline

Serie 9 Review

1. Separation of variables for non-homogeneous problems
   - (a) You could also solve it by finding a particular solution.
   - (b) Non-homogeneous boundary condition –> subtract
2. Conservation of energy
   - (b) $w(x,t) := u(x,t) + F(x)$ with $F''(x) = f(x)$ solves the homogeneous wave equation
3. Multiple choice
   - (a) $k > 1$
4. Extra exercises
   -

Outline

Course Overview

- 1st order PDEs
  - Quasilinear first order PDEs
    - Method of characteristics
    - Conservation laws
- 2nd order PDEs
  - Hyperbolic PDEs
    - Wave equation
    - D'Alembert formula
    - Separation of variables
  - Parabolic PDEs
    - Heat equation
    - Maximum principle
    - Separation of variables
  - Elliptic PDEs
    - **Laplace equation**
    - **Maximum principle**
    - Separation of variables

Outline

From Gauss' Law to Poisson's and Laplace's Equations

Gauss' Law:

$$\nabla \vec{E} = \frac{1}{\epsilon_0}\rho$$

The electric field can be written as the gradient of a scalar potential.

$$\vec{E} = -\nabla V$$

Poisson's Equation:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

In regions where there is no charge, Poisson's equation reduces to Laplace's equation.

$$\Delta V = 0$$

Outline

Laplace's Equation in One Dimension

Suppose V depends on only one variable x. Then Laplace's equation becomes

$$\frac{d^2V}{dx^2} = 0$$

The general solution is a straight line.

$$V(x) = mx + b$$

Notice:

1. $V(x)$ is the average of $V(x+a)$ and $V(x-a)$, for any a:

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$$

Laplace's equation is a kind of averaging instruction; it tells you to assign to the point x the average of the values to the left and to the right of x.

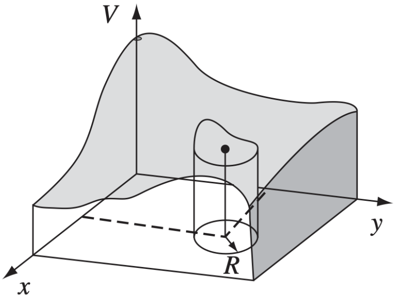
2. Laplace's equation tolerates no local maxima or minima; extreme values of V must occur at the end points. Actually, this is a consequence of (1), for if there were a local maximum, V would be greater at that point than on either side, and therefore could not be the average.

Laplace's Equation in Two Dimensions

If V depends on two variables, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Picture a thin rubber sheet stretched over a box.



Laplace's Equation in Two Dimensions

The height of the tightly stretched rubber membrane satisfies Laplace's Equation.

The one-dimensional analog would be a rubber band stretched between two points, which forms a straight line.

Same properties as in one dimension:

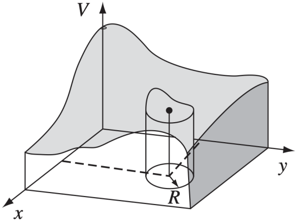
1. The value of V at a point (x, y) is the average of those around the point.

$$V(x, y) = \frac{1}{2\pi R} \oint V\, ds$$

2. V has no local maxima or minima; all extrema occur at the boundaries.
There are no hills, no valleys, just the smoothest conceivable surface.

Laplace's Equation in Two Dimensions

If you put a ping-pong ball on the stretched rubber sheet, it will roll over to one side and fall off.



It will not find a "pocket" somewhere to settle into, for Laplace's equation allows no such dents in the surface.

From a geometrical point of view, just as a straight line is the shortest distance between two points, so a harmonic function in two dimensions minimizes the surface area spanning the given boundary line.

Outline

Poisson's Equation and boundary conditions

Let $D \in \mathbb{R}^2$ an open set and let ∂D be the boundary of D.

Dirichlet problem for Poisson's Equation

$$\begin{cases} \Delta u(x,y) = \rho(x,y), & (x,y) \in D \\ u(x,y) = g(x,y), & (x,y) \in \partial D \end{cases}$$

Neumann problem for Poisson's Equation

$$\begin{cases} \Delta u(x,y) = \rho(x,y), & (x,y) \in D \\ \partial_\nu u(x,y) = g(x,y), & (x,y) \in \partial D \end{cases}$$

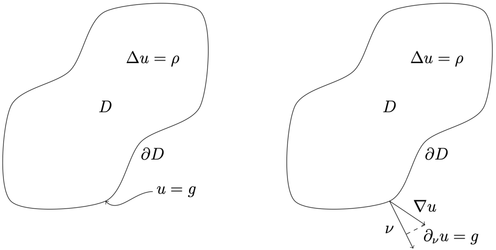


Figure 6.1: Dirichlet and Neumann problems.

Poisson's Equation and boundary conditions

A necessary condition for the existence of a solution to the Neumann problem is

$$\int_{\partial D} g(x(s), y(s))\, ds = \int_D \rho(x, y)\, dx\, dy$$

Proof:

$$\Delta u = \nabla \cdot \nabla u$$

Therefore we can write Poisson's Equation as

$$\nabla \cdot \nabla u = \rho$$

Integrating both sides of the equation over D

$$\int_D \nabla \cdot \nabla u = \int_D \rho$$

Use Gauss' theorem:

$$\int_{\partial D} \nabla u = \int_D \rho$$

Therefore:

$$\int_{\partial D} g = \int_D \rho$$

Laplace's / Poisson's equation describes steady-state / equilibrium

heat flux through the boundary = heat generation inside the domain

Laplace's Equation and harmonic functions

If $\rho = 0$, then Poisson's equation reduces to Laplace's equation, and the condition becomes:

$$\int_{\partial D} \partial_n u = \int_D \nabla \cdot \nabla u = \int_D \Delta u = 0$$

Recall:

A holomorphic function is a complex-valued function that is complex-differentiable (satisfies the Cauchy-Riemann equations) in a neighborhood.

Every holomorphic function can be separated into its real and imaginary parts

$$f(x + iy) = u(x, y) + iv(x, y),$$

and each of these is a harmonic function on $\mathbb{R}^2$

$$\Delta u = \Delta v = 0$$

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Outline

The weak maximum principle

Let D be a bounded domain and let $u(x, y) \in C^2(D) \cap C(\bar{D})$ be a harmonic function in D.

Then the maximum of u in $\bar{D}$ is achieved on the boundary ∂D, namely

$$\max_{\bar{D}} u = \max_{\partial D} u$$

The uniqueness of the Dirichlet Problem

Example 1

Given a bounded domain $D \in \mathbb{R}^2$.

Prove that the Dirichlet problem has at most one solution $u(x, y) \in C^2(D) \cap C(\bar{D})$.

$$\begin{cases} \Delta u(x, y) = \rho(x, y), & (x, y) \in D \\ u(x, y) = g(x, y), & (x, y) \in \partial D \end{cases}$$

Assume by contradiction, that there exist two solutions u_1, u_2.

Then $w := u_1 - u_2$ solves

$$\begin{cases} \Delta w = \Delta u_1 - \Delta u_2 = \rho(x,y) - \rho(x,y) = 0 & \text{in } D \\ w = u_1 - u_2 = g(x,y) - g(x,y) = 0 & \text{on } \partial D \end{cases}$$

w is harmonic in D, and vanishes on ∂D.

From the weak maximum principle, the maximum and minimum of w are zero, which implies $w \equiv 0$ and thus $u_1 \equiv u_2$.

Mean value Principle

Consider a harmonic function u on D and let $B_R(x_0, y_0)$ be a ball of radius R. Then

$$u(x_0, y_0) = \frac{1}{2\pi R} \oint_{\partial B_R(x_0,y_0)} u(x(s), y(s))\, ds$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, d\theta$$

Mean value principle

Example 2

Let $u : D \to \mathbb{R}$ be a solution to Poisson's equation

$$\begin{cases} \Delta u = 1, & \text{in } D \\ u = x^2 + 2y^2 - 1, & \text{on } \partial D \end{cases}$$

where $D = \{x^2 + y^2 < 1\}$.

Compute $u(0,0)$.

Hint: consider the function $v(x,y) = u(x,y) - \frac{y^2}{2}$

$$\begin{cases} \Delta v = \Delta u - \frac{d^2}{dy^2}\left(\frac{y^2}{2}\right) = 1 - 1 = 0 & \text{in } D \\ v = x^2 + 2y^2 - 1 - \frac{y^2}{2} = x^2 + \frac{3}{2}y^2 - 1 & \text{on } \partial D \end{cases}$$

v is harmonic in D, apply mean value principle

$$v(0,0) = \frac{1}{2\pi R}\oint_{\partial B_1} v(x(s), y(s))\,ds = \frac{1}{2\pi}\int_0^{2\pi} v(\cos(\theta), \sin(\theta))\,d\theta$$

$$\to v(0,0) = \frac{1}{2\pi}\int_0^{2\pi} \cos^2(\theta) + \frac{3}{2}\sin^2(\theta) - 1\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos^2(\theta) + \sin^2(\theta) - 1 + \frac{1}{2}\sin^2(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\sin^2(\theta)\,d\theta = \frac{1}{4}$$

$$u(0,0) = v(0,0) - \frac{y^2}{2}\Big|_{y=0} = \frac{1}{4}$$

Mean value principle

Example 2

Let $u : D \to \mathbb{R}$ be a solution to Poisson's equation

$$\begin{cases} \Delta u = 1, & \text{in} \quad D \\ u = x^2 + 2y^2 - 1, & \text{on} \quad \partial D \end{cases}$$

where $D = \{x^2 + y^2 < 1\}$.

What is the maximum of u?

Hint: Consider the function $w(x, y) = u(x, y) + \frac{1-x^2-y^2}{4}$, and note that w is harmonic, $w \geq u$ in D, and $w = u$ on ∂D.

From the weak maximum principle:

$$\max_{\bar{D}} u \leq \max_{\bar{D}} w = \max_{\partial D} w = \max_{\partial D} u$$

Since $x^2+y^2 = 1$ on ∂D

$$\max_{\bar{D}} u = \max_{\partial D} u = \max_{\partial B_1} (x^2+2y^2-1) = \max_{\partial B_1} (x^2+y^2-1)+y^2 = \max_{\partial B_1} y^2 = 1$$

Strong Maximum principle

Let u be a harmonic function in D, an open connected subset of $\mathbb{R}^2$.
If u attains its maximum (or its minimum) at an interior point of D, then u is constant.

Proof:

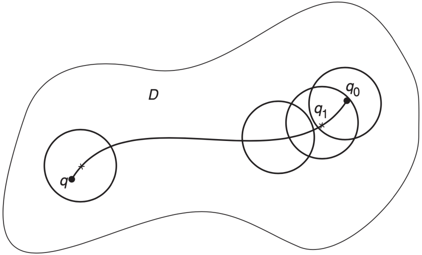

Figure 7.3 A construction for the proof of the strong maximum principle.

Strong Maximum principle
Proof

Assume by contradiction that u obtains its maximum at some interior point q_0.

Let $q \neq q_0$ be an arbitrary point in D.

Consider a disk B_0 around q_0.

Since the average of a set cannot be greater than all the objects of the set, we infer that u is constant in B_0.

It follows that u also reaches its maximal value at q_1.

Thus we obtain that u is constant also in B_1.

We continue in this way until we reach a disk that includes that point q.

We conclude $u(q) = u(q_0)$, and since q is arbitrary, it follows that u is constant in D.

Strong Maximum principle
Example 3

Let $u : D \to \mathbb{R}$ be a solution to the Laplace equation

$$\begin{cases} \Delta u = 0, & \text{in } D \\ u = g, & \text{on } \partial D \end{cases}$$

where $D = \{x^2 + y^2 < 1\}$ and g satisfies $g \geq xy$.

Prove that $u(\frac{1}{2}, \frac{1}{4}) \geq \frac{1}{8}$.

Hint: note that $w = xy$ is harmonic.

$v := u - xy$

$$\begin{cases} \Delta v = \Delta u - \Delta(xy) = 0 & \text{in } D \\ v = u - xy = g - xy & \text{on } \partial D \end{cases}$$

Since $v = g - xy \geq 0$ on ∂D, we deduce that $v \geq 0$ in D.

$\Rightarrow v(\frac{1}{2}, \frac{1}{4}) \geq 0$

$u(\frac{1}{2}, \frac{1}{4}) - xy\big|_{x=\frac{1}{2}, y=\frac{1}{4}} \geq 0$

$u(\frac{1}{2}, \frac{1}{4}) \geq \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

Strong Maximum principle
Example 3

Let $u : D \to \mathbb{R}$ be a solution to the Laplace equation

$$\begin{cases} \Delta u = 0, & \text{in } D \\ u = g, & \text{on } \partial D \end{cases}$$

where $D = \{x^2 + y^2 < 1\}$ and g satisfies $g \geq xy$.

Assume that $u(\frac{1}{2}, \frac{1}{4}) = \frac{1}{8}$. Prove that $g(0, 1) = 0$.

The assumption $u(\frac{1}{2}, \frac{1}{4}) = \frac{1}{8}$ implies that the harmonic function v attains its minimum at $(\frac{1}{2}, \frac{1}{4}) \in D$.

Hence, by the strong maximum principle, v is constant, therefore $v \equiv 0$.

This is equivalent to saying that $u \equiv xy$, and therefore $g \equiv xy$.

In particular $g(0,1) = 0$

Outline

Tips for Serie 10

1. Unique solution
   - Use the hint that $v = u_1 - u_2$.
   - If there were a maximum or minimum in D, what does it imply to Δv?
2. The mean-value principle

$$\frac{1}{\pi R^2} \int_{B_R((x_\circ, y_\circ))} u(x, y)\, dx\, dy = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} u(x_\circ + r\cos\theta, y_\circ + r\sin\theta) r\, d\theta\, dr$$

3. Maximum principle
   - (a) Consult Example 2.
   - (b) $w = u - 3x + y$
4. Multiple choice
   - (a) The necessary condition for the existence of a solution to the Neumann problem
   - (b) Weak maximum principle.
5. Weak maximum principle
   - add w with $\Delta w = 0$.

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Peers found helpful:

1. https://youtu.be/-D4GDdxJrpg

References:

1. Lecture notes on the course website.
2. “An Introduction to Partial Differential Equations” by Yehuda Pinchover and Jacob Rubinstein
3. “Introduction to Electrodynamics” by David J. Griffiths