

9.1. Separation of variables for non-homogeneous problems

Solve the following equations using the method of separation of variables and superposition principle. If the boundary conditions are non-homogeneous, find a suitable function satisfying the boundary conditions, and subtract it from the solution.

(a)

$$\begin{cases} u_t - u_{xx} = t + 2 \cos(2x), & (x, t) \in (0, \pi/2) \times (0, \infty), \\ u_x(0, t) = 0, & t \in (0, \infty), \\ u_x(\pi/2, t) = 0, & t \in (0, \infty), \\ u(x, 0) = 1 + 2 \cos(6x), & x \in [0, \pi/2]. \end{cases}$$

(b)

$$\begin{cases} u_t - u_{xx} = 1 + x \cos(t), & (x, t) \in (0, 1) \times (0, \infty), \\ u_x(0, t) = \sin(t), & t \in (0, \infty), \\ u_x(1, t) = \sin(t), & t \in (0, \infty), \\ u(x, 0) = 1 + \cos(2\pi x), & x \in [0, 1]. \end{cases}$$

Hint: The function $w(x, t) = x \sin(t)$ fulfills the boundary conditions from above.

(c) Mixed Boundary Conditions.

$$\begin{cases} u_t - u_{xx} = \sin(9x/2), & (x, t) \in (0, \pi) \times (0, \infty), \\ u(0, t) = 0, & t \in (0, \infty), \\ u_x(\pi, t) = 0, & t \in (0, \infty), \\ u(x, 0) = \sin(3x/2), & x \in [0, \pi]. \end{cases}$$

9.2. Conservation of energy Suppose $u(x, t)$ is periodic on $(0, \pi)$ and solves $u_{tt} - u_{xx} = f(x)$, for some periodic function f .

(a) Show that if $f \equiv 0$, then the energy

$$E(t) := \frac{1}{2} \int_0^\pi (u_t(x, t))^2 + (u_x(x, t))^2 dx,$$

is conserved, i.e. $E(t) = E(0)$ for all $t > 0$.

(b) Inspired by the homogeneous case, find a similar conserved quantity when $f(x) = \sum_{n=1}^M A_n \sin(nx)$.

9.3. Multiple choice Cross the correct answer(s).

(a) Consider the non-homogeneous heat equation

$$\begin{cases} u_t - u_{xx} = p(t)u, & (x, t) \in (0, \pi) \times (0, \infty), \\ u(0, t) = u(\pi, t) = 0, & t \in (0, \infty), \\ u(x, 0) = f(x), & x \in (0, \pi), \end{cases}$$

where $p(t)$ is a given function of t , and $f(x) = \sum_{n=1}^M A_n \sin(nx) \not\equiv 0$. Then

- ☐ If $p(t) = (k+1)t^k$, $\lim_{t \rightarrow +\infty} e^{-t^{k+1}} u(1, t) = 0$.
- ☐ If $p(t) = (k+1)t^k$, $\lim_{t \rightarrow +\infty} u(1, t) = +\infty$.
- ☐ If $p(t) = \sin(t)$, $\lim_{t \rightarrow +\infty} u(1, t) = +\infty$.
- ☐ If $p(t) = 1$, $\lim_{t \rightarrow +\infty} u(1, t) = A_1 \sin(1)$.

Extra exercises

9.4. Solve the following non-homogeneous problem.

$$\begin{cases} u_t - u_{xx} = -u, & (x, t) \in (0, \pi) \times (0, \infty), \\ u(0, t) = 0, & t \in (0, \infty), \\ u(\pi, t) = 0, & t \in (0, \infty), \\ u(x, 0) = \sin(x), & x \in [0, \pi]. \end{cases}$$